Notes on length and conformal metrics

We recall how to measure the Euclidean distance of an arc in the plane. Let $\alpha : [a, b] \longrightarrow \mathbb{R}^2$ be a smooth ($C^1$) arc. That is $\alpha(t) = (x(t), y(t))$ where $x(t)$ and $y(t)$ are smooth real valued functions. Then the length of $\alpha$ is the integral

$$|\alpha| = \int_a^b |\alpha'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$ 

Note that if $\alpha$ is only piecewise smooth we can still define $|\alpha|$. In particular if $\alpha$ is piecewise smooth the derivative $\alpha'$ will be defined at all but finitely many points in the interval $[a, b]$ so the above integral still makes sense.

Many formulas become simpler by using complex notation. That is we think of $\alpha$ as a map to $\mathbb{C}$ by setting $\alpha(t) = x(t) + iy(t)$. Then $\alpha'(t) = x'(t) + iy'(t)$ is also a complex number. Thought of as a complex number the absolute value of $\alpha(t)$ gives us the same answer: $|\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$. Note that the using the books notation we have

$$|\alpha| = \int_\alpha |dz| = \int_a^b |\alpha'(t)| dt.$$

Let $\Omega$ be an open subset of $\mathbb{R}^2$ that contains the image of $\alpha$ and let $f : \Omega \longrightarrow \mathbb{R}^2$ be a smooth function. We then have a new path define by $\bar{\alpha} = f \circ \alpha$. To calculate the length of $\bar{\alpha}$ we use the chain rule. In particular, if $f(x, y) = (u(x, y), v(x, y))$ then $\bar{\alpha}'(t)$, written as a column vector, is

$$\bar{\alpha}'(t) = \left( \begin{array}{c} u_x(\alpha(t)) \\
 v_x(\alpha(t)) \end{array} \right) \left( \begin{array}{c} x'(t) \\
 y'(t) \end{array} \right).$$

We can think of $f$ has a complex function by setting $z = x + iy$ and $f = u + iv$. If $f$ is holomorphic we really see the advantage of using complex notation. The Cauchy-Riemann equations tell us that $u_x = v_y$ and $v_x = -u_y$. Furthermore the complex derivative of $f$ is $f' = u_x + iv_x$. If we treat $\bar{\alpha}'(t)$ as a complex number we see that

$$\bar{\alpha}' = u_x x' - v_x y' + i(v_x x' + u_x y') = (u_x + iv_x)(x' + iy').$$

That is we have $\bar{\alpha}'(t) = f'(\alpha(t))\alpha'(t)$. This gives a very simple formula for the length of $\bar{\alpha}$:

$$|\bar{\alpha}| = \int_a^b |f'(\alpha(t))||\alpha'(t)| dt.$$

We say that $f$ is an isometry of the Euclidean metric if the length of every path $\alpha$ is equal to the length of the path $\bar{\alpha} = f \circ \alpha$. Clearly $f$ is an isometry if $|f'| \equiv 1$. In fact
it is not hard to see that this is also a necessary condition since if $|f'(z)| < 1$ at $z$ then
by continuity this will be true in a neighborhood $U$ of $z$. For any path $\alpha$ whose image
is contained in $U$ we will then have that $\bar{\alpha}$ is shorter than $\alpha$. We can make a similar
argument if $|f'(z)| > 1$ at $z$.

In a homework problem we saw that any holomorphic function that had a constant
absolute value must be constant. In class we will soon see that the derivative, $f'$, of
a holomorphic function is also holomorphic. For now we take this as an assumption.
Therefore if $|f'(z)| \equiv 1$ then $f'(z) \equiv c$ where $|c| = 1$ and $f$ must be of the form
$f(z) = cz + d$ where $d$ is an arbitrary complex number.

It is often useful to use alternative definitions of a distance. In particular if $\Omega$ is again
an open subset of $\mathbb{R}^2$ let $\lambda : \Omega \rightarrow \mathbb{R}$ be a positive function. We can define the length of $\alpha$ with respect to $\lambda$ by

$$|\alpha|_{\lambda} = \int_{a}^{b} |\alpha'(t)| \lambda(\alpha(t)) dt.$$

If we have two different metrics defined by functions $\lambda$ and $\rho$ we can then discuss
whether $f$ is an isometry from the $\lambda$-metric to the $\rho$-metric. To measure the length $\bar{\alpha}$ in the $\rho$-metric we have the formula

$$|\bar{\alpha}|_{\rho} = \int_{a}^{b} |\bar{\alpha}'(t)| \rho(\bar{\alpha}(t)) dt = \int_{a}^{b} |f'(\alpha(t))||\alpha'(t)| \rho(f(\alpha(t))) dt.$$

For this to be the same as the $\lambda$-length of $\alpha$ for all paths $\alpha$ we need to have

$$|f'(\alpha(t))| \rho(f(\alpha(t))) = \lambda(\alpha(t))$$

or

$$|f'(z)| \rho(f(z)) = \lambda(z).$$

Note that this formula gives us a way for defining a metric. In particular if $\rho \equiv 1$ then
the $\rho$-metric is just the standard Euclidean metric. If we define $\lambda$ by setting

$$\lambda(z) = |f'(z)|$$

then $f$ will be an isometry from the $\lambda$-metric to the Euclidean metric. If we define $\lambda$ by

$$\lambda(z) = |f'(z)| \rho(f(z))$$

then $f$ is an isometry from $\lambda$-metric to the $\rho$-metric.

One very useful metric that we will work with is the hyperbolic metric. It is defined
on the upper half plane of $\mathbb{C}$ which we define as

$$\mathbb{H}^2 = \{ z \in \mathbb{C} : \text{Im } z > 0 \}.$$
The hyperbolic metric is \( \lambda_{\mathbb{H}^2}(z) = \frac{1}{\Im z} \). The isometries of the hyperbolic metric are linear fractional transformations that preserve the upper half plane. Namely let

\[
T(z) = \frac{az + b}{cz + d}
\]

where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc = 1 \). Then

\[
T'(z) = \frac{1}{(cz + d)^2}.
\]

We also need to calculate \( \Im T(z) \):

\[
2\Im T(z) = T(z) - \overline{T(z)} = \frac{az + b}{cz + d} - \frac{a\overline{z} + b}{c\overline{z} + d} = \frac{(az + b)(c\overline{z} + d) - (a\overline{z} + b)(cz + d)}{|cz + d|^2} = \frac{(ad - bc)(z - \overline{z})}{|cz + d|^2}.
\]

and therefore

\[
\Im T(z) = \frac{\Im z}{|cz + d|^2}.
\]

We then have

\[
|T'(z)|\lambda_{\mathbb{H}^2}(T(z)) = \frac{1}{|cz + d|^2} \frac{1}{\Im T(z)} = \frac{1}{|cz + d|^2} \frac{|cz + d|^2}{\Im z} = \frac{1}{\Im z} = \lambda_{\mathbb{H}^2}(z)
\]

so \( T(z) \) is an isometry for the hyperbolic metric.

We can use the metric \( \lambda \) to define a distance function on the region \( \Omega \). Let \( \mathcal{P}(z_0, z_1) \) be the set of piecewise smooth paths in \( \Omega \) from \( z_0 \) to \( z_1 \). We then define

\[
d_\lambda(z_0, z_1) = \inf_{\gamma \in \mathcal{P}(z_0, z_1)} |a|_\lambda.
\]

It is easy to check that \( d_\lambda \) satisfies the properties of a distance function:
1. Clearly $d_\lambda(z_0, z_1) = d_\lambda(z_1, z_0)$ since by reversing directions any path from $z_0$ to $z_1$ becomes a path from $z_1$ to $z_0$ of the same length.

2. It is also easy to check the triangle inequality. (Here it is important that we are allowing piecewise smooth paths.) If we concatenate a path from $z_0$ to $z_1$ with a path from $z_1$ to $z_2$ we obtain a path from $z_0$ to $z_2$. In particular if there is a path of length $\ell_0$ from $z_0$ to $z_1$ and a path of length $\ell_1$ from $z_1$ to $z_2$ then there is a path of length $\ell_0 + \ell_1$ from $z_0$ to $z_1$. This implies that

$$d_\lambda(z_0, z_2) \leq d(z_0, z_1) + d(z_1, z_2).$$

3. Finally we need to see that $d(z_0, z_1) = 0$ iff $z_0 = z_1$. The function $\lambda$ is continuous and positive so for any $z_0$ there is an $\epsilon > 0$ and an $r > 0$ so that on the Euclidean disk of radius $r$ such that $\lambda > \epsilon$ on the disk. Let $\alpha$ be a path from $z_0$ to $z_1$. If $\alpha$ is contained in this Euclidean disk then $|\alpha|_\lambda > \epsilon|\alpha| \geq \epsilon d(z_0, z_1) > 0$ if $z_0 \neq z_1$. If $\alpha$ is not contained in the disk there is a sub-path $\alpha'$ connecting $z_0$ to the boundary of the disk so $|\alpha|_\lambda \geq |\alpha'|_\lambda \geq \epsilon r > 0$. In particular if $z_1 \neq z_1$ is in the disk then $d_\lambda(z_0, z_1) \geq \epsilon d(z_0, z_1) > 0$ and if $z_1$ is not in the disk then $d(z_0, z_1) \geq \epsilon r > 0$ so $d(z_0, z_1) > 0$ if $z_0 \neq z_1$. It is clear that $d(z_0, z_1) = 0$ if $z_1 = z_0$.

The distance function makes $(\Omega, d_\lambda)$ into a metric space and we will be able to use all the properties of metric spaces to study it. We also note if $\rho \leq \lambda$ defines another metric on $\Omega$ then $d_\rho(z_0, z_1) \leq d_\lambda(z_0, z_1)$ for all points $z_0, z_1 \in \Omega$.

Problems

1. Let $\Delta$ be the unit disk in $\mathbb{C}$. Construct a linear fraction transformation $S : \mathbb{C} \longrightarrow \mathbb{C}$ that takes $\Delta$ to the upper half plane.

2. Define a metric $\rho$ on $\Delta$ by the formula

$$\rho(z) = \frac{2}{1 - |z|^2}.$$  

Show that $S$ is an isometry from the $\rho$-metric to the hyperbolic metric $\lambda_{\mathbb{H}^2}$. In particular, the metric $\rho$ on $\Delta$ is another representation of the hyperbolic metric. To emphasize this we write $\rho$ as $\rho_{\mathbb{H}^2}$.

3. The $f(z) = z^2$ take $\Delta$ to itself. Show that for any two points $z_0 \neq z_1$ in $\Delta$ we have

$$d_{\rho_{\mathbb{H}^2}}(f(z_0), f(z_1)) \leq d_{\rho_{\mathbb{H}^2}}(z_0, z_1).$$
4. Define a metric on $\mathbb{C}$ by $\sigma(z) = \frac{2}{1+|z|^2}$. Given a point $z \in \mathbb{C}$ find a linear fractional transformation $R$ with $R(0) = z$, $R(\infty) = -\frac{1}{z}$ and such that $R$ is an isometry for $\sigma$-metric.

Comments: Problem 3 is an example of a very important and much more general phenomenon. In particular any holomorphic map that takes $\Delta$ into itself will be a contraction of the hyperbolic metric. This is essentially the Schwarz Lemma which we will (soon!) prove in class.