Notes on length and conformal metrics

We recall how to measure the Euclidean distance of an arc in the plane. Let $\alpha : [a,b] \longrightarrow \mathbb{R}^2$ be a smooth (C^1) arc. That is $\alpha(t) = (x(t), y(t))$ where $x(t)$ and $y(t)$ are smooth real valued functions. Then the length of α is the integral

$$|\alpha| = \int_a^b |\alpha'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt.$$

Note that if α is only piecewise smooth we can still define $|\alpha|$. In particular if α is piecewise smooth the derivative α' will be defined at all but finitely many points in the interval $[a,b]$ so the above integral still makes sense.

Many formulas become simpler by using complex notation. That is we think of α as a map to $\mathbb{C}$ by setting $\alpha(t) = x(t) + \imath y(t)$. Then $\alpha'(t) = x'(t) + \imath y'(t)$ is also a complex number. Thought of as a complex number the absolute value of $\alpha(t)$ gives us the same answer: $|\alpha'(t)| = \sqrt{x'(t)^2 + \imath y'(t)}$. Note that the using the books notation we have

$$|\alpha| = \int_\alpha |dz| = \int_a^b |\alpha'(t)|dt.$$

Let Ω be an open subset of $\mathbb{R}^2$ that contains the image of α and let $f : \Omega \longrightarrow \mathbb{R}^2$ be a smooth function. We then have a new path define by $\bar{\alpha} = f \circ \alpha$. To calculate the length of $\bar{\alpha}$ we use the chain rule. In particular, if $f(x,y) = (u(x,y), v(x,y))$ then $\bar{\alpha}'(t)$, written as a column vector, is

$$\bar{\alpha}'(t) = \begin{pmatrix} u_x(\alpha(t)) & u_y(\alpha(t)) \\ v_x(\alpha(t)) & v_y(\alpha(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

We can think of f has a complex function by setting $z = x + \imath y$ and $f = u + \imath v$. If f is holomorphic we really see the advantage of using complex notation. The Cauchy-Riemann equations tell us that $u_x = v_y$ and $v_x = -u_y$. Furthermore the complex derivative of f is $f' = u_x + \imath v_x$. If we treat $\bar{\alpha}'(t)$ as a complex number we see that

$$\begin{aligned} \bar{\alpha}' &= u_x x' - v_x y' + \imath(v_x x' + u_x y') \\ &= (u_x + \imath v_x)(x' + \imath y'). \end{aligned}$$

That is we have $\bar{\alpha}'(t) = f'(\alpha(t))\alpha'(t)$. This gives a very simple formula for the length of $\bar{\alpha}$:

$$|\bar{\alpha}| = \int_a^b |f'(\alpha(t))||\alpha'(t)|dt.$$

We say that f is an *isometry* of the Euclidean metric if the length of every path α is equal to the length of the path $\bar{\alpha} = f \circ \alpha$. Clearly f is an isometry if $|f'| \equiv 1$. In fact it is not hard to see that this is also a necessary condition since if $|f'(z)| < 1$ at z then by continuity this will be true in a neighborhood U of z. For any path α whose image is contained in U we will then have that $\bar{\alpha}$ is shorter than α. We can make a similar argument if $|f'(z)| > 1$ at z.

In a homework problem we saw that any holomorphic function that had a constant absolute vale must be constant. In class we will soon see that the derivative, f', of a holomorphic function is also holomorphic. For now we take this as an assumption. Therefore if $|f'(z)| \equiv 1$ then $f'(z) \equiv c$ where $|c| = 1$ and f must be of the form $f(z) = cz + d$ where d is an arbitrary complex number.

It is often useful to use alternative definitions of a distance. In particular if Ω is again an open subset of $\mathbb{R}^2$ let $\lambda : \Omega \longrightarrow \mathbb{R}$ be a positive function. We can define the length of α with respect to λ by

$$|\alpha|_\lambda = \int_a^b |\alpha'(t)|\lambda(\alpha(t))dt.$$

If we have two different metrics defined by functions λ and ρ we can then discuss whether f is an isometry from the λ-metric to the ρ-metric. To measure the length $\bar{\alpha}$ in the ρ-metric we have the formula

$$|\bar{\alpha}|_\rho = \int_a^b |\bar{\alpha}'(t)|\rho(\bar{\alpha}(t))dt = \int_a^b |f'(\alpha(t))||\alpha'(t)|\rho(f(\alpha(t)))dt.$$

For this to be the same as the λ-length of α for all paths α we need to have

$$|f'(\alpha(t))|\rho(f(\alpha(t))) = \lambda(\alpha(t))$$

or

$$|f'(z)|\rho(f(z)) = \lambda(z).$$

Note that this formula gives us a way for defining a metric. In particular if $\rho \equiv 1$ then the ρ-metric is just the standard Euclidean metric. If we define λ by setting

$$\lambda(z) = |f'(z)|$$

then f will be an isometry from the λ-metric to the Euclidean metric. If we define λ by

$$\lambda(z) = |f'(z)|\rho(f(z))$$

then f is an isometry from λ-metric to the ρ-metric.

One very useful metric that we will work with is the hyperbolic metric. It is defined on the upper half plane of $\mathbb{C}$ which we define as

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

The hyperbolic metric is $\lambda_{\mathbb{H}^2}(z) = \frac{1}{\operatorname{Im} z}$. The isometries of the hyperbolic metric are linear fractional transformations that preserve the upper half plane. Namely let

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Then

$$T'(z) = \frac{1}{(cz+d)^2}.$$

We also need to calculate $\operatorname{Im} T(z)$:

$$\begin{aligned}
2\imath \operatorname{Im} T(z) &= T(z) - \overline{T(z)} \\
&= \frac{az+b}{cz+d} - \overline{\left(\frac{az+b}{cz+d}\right)} \\
&= \frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d} \\
&= \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2} \\
&= \frac{(ad-bc)(z-\overline{z})}{|cz+d|^2} \\
&= \frac{2\imath \operatorname{Im} z}{|cz+d|^2}
\end{aligned}$$

and therefore

$$\operatorname{Im} T(z) = \frac{\operatorname{Im} z}{|cz+d|^2}.$$

We then have

$$\begin{aligned}
|T'(z)|\lambda_{\mathbb{H}^2}(T(z)) &= \frac{1}{|cz+d|^2}\frac{1}{\operatorname{Im} T(z)} \\
&= \frac{1}{|cz+d|^2}\frac{|cz+d|^2}{\operatorname{Im} z} \\
&= \frac{1}{\operatorname{Im} z} \\
&= \lambda_{\mathbb{H}^2}(z)
\end{aligned}$$

so $T(z)$ is an isometry for the hyperbolic metric.

We can use the metric λ to define a distance function on the region Ω. Let $\mathcal{P}(z_0, z_1)$ be the set of piecewise smooth paths in Ω from z_0 to z_1. We then define

$$d_\lambda(z_0, z_1) = \inf_{\gamma \in \mathcal{P}(z_0,z_1)} |\alpha|_\lambda.$$

It is easy to check that d_λ satisfies the properties of a distance function:

1. Clearly $d_\lambda(z_0, z_1) = d_\lambda(z_1, z_0)$ since by reversing directions any path from z_0 to z_1 becomes a path from z_1 to z_0 of the same length.

2. It is also easy to check the triangle inequality. (Here it is important that we are allowing piecewise smooth paths.) If we concatenate a path from z_0 to z_1 with a path from z_1 to z_1 we obtain a path from z_0 to z_2. In particular if there is a path of length ℓ_0 from z_0 to z_1 and a path of length ℓ_1 from z_1 to z_2 then there is a path of length $\ell_0 + \ell_1$ from z_0 to z_1. This implies that

$$d_\lambda(z_0, z_2) \leq d(z_0, z_1) + d(z_1, z_2).$$

3. Finally we need to see that $d(z_0, z_1) = 0$ iff $z_0 = z_1$. The function λ is continuous and positve so for any z_0 there is an $\epsilon > 0$ and an $r > 0$ so that on the Euclidean disk of radius r such that $\lambda > \epsilon$ on the disk. Let α be a path from z_0 to z_1. If α is contained in this Euclidean disk then $|\alpha|_\lambda > \epsilon|\alpha| \geq \epsilon d(z_0, z_1) > 0$ if $z_0 \neq z_1$. If α is not contained in the disk there is a sub-path α' connecting z_0 to the boundary of the disk so $|\alpha|_\lambda \geq |\alpha'|_\lambda \geq \epsilon r > 0$. In particular if $z_1 \neq z_1$ is in the disk then $d_\lambda(z_0, z_1) \geq \epsilon d(z_0, z_1) > 0$ and if z_1 is not in the disk then $d(z_0, z_1) \geq \epsilon r > 0$ so $d(z_0, z_1) > 0$ if $z_0 \neq z_1$. It is clear that $d(z_0, z_1) = 0$ if $z_1 = z_0$.

The distance function makes (Ω, d_λ) into a *metric space* and we will be able to use all the properties of metric spaces to study it. We also note if $\rho \leq \lambda$ defines another metric on Ω then $d_\rho(z_0, z_1) \leq d_\lambda(z_0, z_1)$ for all points $z_0, z_1 \in \Omega$.

Problems

1. Let Δ be the unit disk in $\mathbb{C}$. Construct a linear fraction transformation $S : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ that takes Δ to the upper half plane.

2. Define a metric ρ on Δ by the formula

$$\rho(z) = \frac{2}{1 - |z|^2}.$$

Show that S is an isometry from the ρ-metric to the hyperbolic metric $\lambda_{\mathbb{H}^2}$. In particular, the metric ρ on Δ is another representation of the hyperbolic metric. To emphasize this we write ρ as $\rho_{\mathbb{H}^2}$.

3. The $f(z) = z^2$ take Δ to itself. Show that for any two points $z_0 \neq z_1$ in Δ we have

$$d_{\rho_{\mathbb{H}^2}}(f(z_0), f(z_1)) \leq d_{\rho_{\mathbb{H}^2}}(z_0, z_1).$$

4. Define a metric on $\mathbb{C}$ by $\sigma(z) = \frac{2}{1+|z|^2}$. Given a point $z \in \mathbb{C}$ find a linear fractional transformation R with $R(0) = z$, $R(\infty) = -\frac{1}{\bar{z}}$ and such that R is an isometry for σ-metric.

Comments: Problem 3 is an example of a very important and much more general phenomenom. In particular any holomorphic map that takes Δ into itself will be a contraction of the hyperbolic metric. This is essentially the Schwarz Lemma which we will (soon!) prove in class.