• **Product rule:** If \( F(x) = f(x)g(x) \) then
\[
F'(x) = f'(x)g(x) + f(x)g'(x).
\]

• **Quotient rule:** If \( F(x) = \frac{f(x)}{g(x)} \) then
\[
F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\]

• **Chain rule:** If \( F(x) = f(g(x)) \) then
\[
F'(x) = f'(g(x))g'(x).
\]

• **Power rule:** If \( F(x) = f(x)^n \) then
\[
F'(x) = nf(x)^{n-1}f'(x).
\]

• **Quadratic formula:** If \( ax^2 + bx + c = 0 \) then
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

• **Integrals:**
\[
\int f'(x)f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C \quad \text{if } n \neq -1;
\]
\[
\int f'(x)f(x)^{-1}\,dx = \ln|f(x)| + C;
\]
\[
\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C.
\]

• **Elasticity:** If \( q \) is the quantity and \( p \) is the price then the elasticity is
\[
\frac{-p}{q}\frac{dq}{dp}.
\]

• **Present value:** If \( f(t) \) is a revenue stream and \( r \) the interest then the future value of the stream after \( T \) years is
\[
\int_0^T f(t)e^{-rt}\,dt.
(1) Find the derivative $\frac{dy}{dx}$ if

(a) $y = xe^{x^2}$

$f(x) = x \quad \Rightarrow \quad f'(x) = 1$

$g(x) = e^{x^2} \quad \Rightarrow \quad g'(x) = 2xe^{x^2}$

$h(x) = e^x \quad \Rightarrow \quad h'(x) = e^x$

$r(x) = x^2 \quad \Rightarrow \quad r'(x) = 2x$

\[
\frac{dy}{dx} = 1 \cdot e^{x^2} + x \cdot (2xe^{x^2})
\]
(b) \( y = \ln(x^2 + 3) \) = \( f(g(x)) \)

\[
\begin{align*}
  f(x) &= \ln x \\
  f'(x) &= \frac{1}{x} \\
  g(x) &= x^2 + 3 \\
  g'(x) &= 2x
\end{align*}
\]

\[
\frac{dy}{dx} = \frac{1}{x^2 + 3} (2x)
\]
(c) \( y = \frac{1}{\sqrt{x^2 + 1}} = (x^2 + 1)^{-1/2} = f(x)^n \)

\[ f(x) = x^2 + 1 \]

\[ f'(x) = 2x \]

\[ n = -\frac{1}{2} \]

\[ \frac{dy}{dx} = -\frac{1}{2} (x^2 + 1)^{-1/2 - 1} (2x) \]

\[ = -\frac{x}{(x^2 + 1)^{3/2}} \]
(d) \( y = \frac{e^{\sqrt{x}}}{1+x} = \frac{f(x)}{g(x)} \)

\[ f(x) = h(u(x)) \]
\[ h(x) = e^x \quad \Rightarrow \quad h'(x) = e^x \]
\[ u(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \quad u'(x) = \frac{1}{2} x^{-1/2} \]
\[ f'(x) = e^{\sqrt{x}} \left( \frac{1}{2} x^{-1/2} \right) = \frac{e^{u(x)}}{2\sqrt{x}} \]
\[ g(x) = 1+x \quad \Rightarrow \quad g'(x) = 1 \]

\[ \frac{dy}{dx} = \frac{e^{u(x)} (1+x) - e^{\sqrt{x}} \cdot 1}{(1+x)^2} \]
(e) $y = \ln(x(x^2 + 1))$

$= \ln x + \ln(x^2 + 1)$

$= \ln x + f(g(x))$

$g(x) = x^2 + 1, \quad g'(x) = 2x$

$f(x) = \ln x, \quad f'(x) = \frac{1}{x}$

$F'(x) = \frac{1}{x^2 + 1} (2x)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{2x}{x^2 + 1}$

\[
\frac{x^2 + 1 + 2x}{x(x^2 + 1)}
\]
\( f(x) = e^{3x^2+1} \)

\[ f(x) = e^x \quad f'(x) = e^x \]

\[ g(x) = 3x^2 + 1 \quad g'(x) = 6x \]

\[ \frac{dy}{dx} = e^{3x^2+1} (6x) \]
(2) Find the second derivative of \( f(x) = \frac{x}{3x+1} \).

\[
\begin{align*}
  u(x) &= x & u'(x) &= 1 \\
  v(x) &= 3x+1 & v'(x) &= 3 \\
  f'(x) &= \frac{1 \cdot (3x+1) - x \cdot 3}{(3x+1)^2} = \frac{1}{(3x+1)^2} = (3x+1)^{-2} = g(x)^{-2} \\
  g(x) &= 3x+1 & g'(x) &= 3 \\
  f''(x) &= -2 \cdot (3x+1)^{-3} (3) = -\frac{6}{(3x+1)^3}
\end{align*}
\]
(3) Find the equation of the tangent line to \( y^3 = x^2 - 3 \) at \((x, y) = (-2, 1)\).

\[
3y^2 \frac{dy}{dx} = 2x
\]

At \((x, y) = (-2, 1)\)

\[
3 \cdot 1 \frac{dy}{dx} = 2(-2) \Rightarrow \frac{dy}{dx} = -\frac{4}{3}
\]

Point-slope formula:

\[
y - 1 = -\frac{4}{3} (x + 2)
\]
(4) Let \( f(x) = \frac{4x^2+1}{3x^2+1} \). The first derivative of \( f \) is 
\[
 f'(x) = \frac{2x}{(3x^2+1)^2}
\]
and the second derivative is 
\[
 f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}.
\]
(a) Find the critical values of $f$.

\[
\frac{f}{I}x_1 = \frac{4x+1}{3x^2+1}
\]

\[
\frac{f'}{f'}(x) = \frac{2x}{3x^2+1}
\]

\[
\frac{f''}{f''}(x) = \frac{2(-9x)}{(3x^2+1)^2}
\]

\[f'(x) = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0\]

\[f'(x) \text{ is undefined} \Leftrightarrow (3x^2+1)^2 = 0 \Leftrightarrow f'(x) \text{ is always defined}\]

C.V. \hspace{1cm} x = 0
(b) Find the intervals where $f$ is increasing and the intervals where $f$ is decreasing.

The only critical value is $x = 0$

so the intervals are $(-\infty, 0)$ &

$(0, \infty)$. We need to decide on which

of these intervals the function is

increasing or decreasing.

As $-1 \in (-\infty, 0)$ & $f'(-1) = \frac{-2}{4^2} < 0$

$f$ is decreasing on $(-\infty, 0)$

As $1 \in (0, \infty)$ & $f'(1) = \frac{2}{4^2} > 0$

$f$ is increasing on $(0, \infty)$. 

\[
\frac{f(x)}{3x^2} = 2x \quad \frac{f'(x)}{2x^2} = \frac{2x}{3x^2+1}^2
\]

\[
\frac{f''(x)}{3x^2} = \frac{2(1-4x^2)}{(3x^2+1)^3}
\]
\( f'(x) = \frac{4x^2}{x^4 - 1} \)

\( f''(x) = \frac{2(1 - 9x^2)}{(3x^4 - 1)^2} \)

We first find where \( f''(x) = 0 \).

This occurs when \( 2(1 - 9x^2) = 0 \)

\[ \Rightarrow 9x^2 = 1 \Rightarrow x = \pm \frac{1}{3} \]

So our intervals are \((-\infty, -\frac{1}{3}), \left(\frac{1}{3}, \frac{1}{3}\right)\) and \((\frac{1}{3}, +\infty)\). Choosing points in each interval we see that \( f''(-1) = \frac{2(-8)}{4\sqrt{3}} < 0 \)

\( f''(0) = \frac{2}{1^3} > 0 \) & \( f''(1) = \frac{2(-8)}{4\sqrt{3}} < 0 \)

We have \( f \) is concave up on \((-\frac{1}{3}, \frac{1}{3})\) & concave down on \((-\infty, -\frac{1}{3}) \) & \((\frac{1}{3}, +\infty)\).
(d) Find $x$-values for any inflection points for $f$.

The inflection pts are when concavity changes so they are $x = \pm \frac{1}{3}$. 

\[
\frac{f(x)}{f''(x)} = \frac{4x^4}{2x^3},
\]

\[
\frac{f'(x)}{f''(x)} = \frac{3x}{2x^3}.
\]

\[
\frac{f''(x)}{f''(x)} = \frac{2(1-x^2)}{(3x^3+1)^2}.
\]
(e) Find the $x$-values of any relative maxima and minima. Make sure you clear which values correspond to maxima and which to minima.

$$f'(x) = \frac{4x^4+1}{2x^3}$$

$$f''(x) = \frac{2x}{3x^2+1}$$

$$f'''(x) = \frac{2(1-x^2)}{(3x^2+1)^2}$$

The relative extrema occur at the critical values. Here $x=0$ is the only critical value. We need to determine if it is a relative max or min.

Note that $f''(0) > 0$, so by the 2nd derivative test $x=0$ is a relative min.
To find the horizontal asymptote we take $\lim_{x \to \infty} \frac{4x^2 + 1}{3x^2} = \lim_{x \to \infty} \frac{4x^2}{3x^2} = \frac{4}{3}.$

So the horizontal asymptote is $y = \frac{4}{3}.$

Vertical asymptotes occur when the denominator of $f(x)$ is zero. Since $3x^2 + 1 > 1$ for all values of $x$ there are no vertical asymptotes.
(5) If \( f(x, y) = \frac{x}{y - x} \), determine the following:

\[
\frac{\partial f}{\partial x} = \frac{u(x, y)}{v(x, y)} \quad \text{where}
\]

\[
u(x, y) = x \quad \text{and} \quad v(x, y) = y - x .
\]

Taking partial derivatives we have

\[
\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = -1 \quad \text{so}
\]

\[
\frac{\partial f}{\partial x} = \frac{\frac{\partial u}{\partial x} \cdot v - u \cdot \frac{\partial v}{\partial x}}{v^2} = \frac{1 \cdot (y - x) - x \cdot (-1)}{(y - x)^2} = \frac{y}{(y - x)^2}
\]

(a) \( \frac{\partial f}{\partial x} \)
(b) $\frac{\partial^2 f}{\partial x \partial y}$

As $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ we need to take the partial derivative of $\frac{\partial f}{\partial x} = \frac{y}{(y-x)^2}$ with respect to $y$.

We have $\frac{y}{(y-x)^2} = \frac{u(x,y)}{v(x,y)}$ where $(y-x)^2 = v(x,y)$

$u(x,y) = y$ & $v(x,y) = (y-x)^2$ so

$\frac{\partial u}{\partial y} = 1$ & $\frac{\partial v}{\partial y} = 2(y-x)$ (1).

Therefore

$\frac{\partial^2 f}{\partial y \partial x} = \frac{1 \cdot (y-x)^2 - y \cdot 2(y-x)}{(y-x)^4}$
(6) Compute the following integrals:

We use the log rule with

\[ u = x^2 + 1 \quad \text{and} \quad u' = 2x \]

Since \( \int \frac{3x}{x^2 + 1} \, dx = \frac{3}{2} \int \frac{2x}{x^2 + 1} \, dx \)

we have

\[ \int \frac{3x}{x^2 + 1} \, dx = \frac{3}{2} \left( \ln |x^2 + 1| + C \right). \]
(b) \( \int \frac{x}{(x^2 + 2)^2} \, dx = \int x \cdot (x^2 + 2)^{-2} \, dx. \)

Let \( u = x^2 + 2 \) & \( u = -2 \). So
\( u' = 2x \). Then by the power rule
\[
\int 2x \cdot (x^2 + 2)^{-2} \, dx = \frac{\left(x^2 + 2\right)^{-1}}{-1} + C
\]

& since
\[
\int \frac{x}{(x^2 + 2)^2} \, dx = \frac{1}{2} \int 2x \cdot (x^2 + 2)^{-2} \, dx
\]

we have
\[
\int \frac{x}{(x^2 + 2)^{3/2}} \, dx = \frac{1}{2} \left( \frac{\left(x^2 + 2\right)^{-1}}{-1} + C \right)
\]
(c) \[ \int 8x\sqrt{x^2 - 1} \, dx = \int 8x(x^2-1)^{\frac{1}{2}} \, dx \]

\[ 4\int 2x(x^2-1)^{\frac{1}{2}} \, dx \]

\[ = 4\left( \frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}} + C \right) \]

Let \( u = x^2 - 1 \) so \( u' = 2x \)
& \( n = \frac{1}{2} \). To apply the power rule we see that for the red integral to equal the black we need to multiply by \( 4 \). Therefore

\[ \int 8x\sqrt{x^2 - 1} \, dx = 4\left( \frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}} + C \right) . \]
\[(d) \int_{0}^{1} \frac{x}{e^{x^2}} \, dx = \int_{0}^{1} x e^{-x^2} \, dx\]

\[y = -x^2 \quad u' = -2x\]

so \[\int_{0}^{1} \frac{x}{e^{x^2}} \, dx = -\frac{1}{2} \int_{-2}^{0} e^{-x^2} \, dx\]

\[= -\frac{1}{2} \left( e^{-x^2} \bigg|_{-2}^{0} \right) = -\frac{1}{2} \left( e^{-1} - e^{0} \right)\]

\[= \frac{1}{2} \left( 1 - \frac{1}{2} \right).\]
(e) \[ \int_{0}^{4} \frac{x}{\sqrt{x^2 + 9}} \, dx = \int_{0}^{4} x \cdot (x^2 + 9)^{-\frac{1}{2}} \, dx \]

\[ u = x^2 + 9 \quad \text{and} \quad u' = 2x \quad \Rightarrow \]

\[ \int_{0}^{4} \frac{x}{\sqrt{x^2 + 9}} \, dx = \frac{1}{2} \int_{0}^{4} 2x \cdot (x^2 + 9)^{-\frac{1}{2}} \, dx \]

\[ = \frac{1}{2} \left( \frac{(x^2 + 9)^{\frac{1}{2}}}{\frac{9}{2}} \right) \bigg|_{0}^{4} \]

\[ = \frac{1}{2} \left( \frac{16 + 9}{\frac{9}{2}} - \frac{9}{\frac{9}{2}} \right) = 5 - 3 = 2 \]
(7) Suppose that a donor wishes to provide a cash gift to the University that will generate a continuous income stream with an annual rate of flow at time \( t \) given by \( f(t) = 80,000 \) per year. If the annual interest rate of 4% compounded continuously, find the capital value of this income stream.

Find the present value of the income stream after 10 years.

\[
P V = \int_0^{10} f(t) e^{-0.04t} \, dt
\]

\[
= \int_0^{10} 8000 \, e^{-0.04t} \, dt
\]

\[
= \frac{8000}{-0.04} \left[ e^{-0.04t} \right]_0^{10}
\]

\[
= \frac{8000}{-0.04} \left( e^{-0.4} - e^0 \right)
\]

\[
= 200,000 \left( 1 - \frac{1}{e^{0.4}} \right).
\]
(8) A company has an income stream of \( f(t) = 7000e^{-0.03t} \) in dollars per year. Find the present value of the income stream over the next 10 years if it is invested at a rate of 2% a year compounded continuously. Also find the capital value of the income stream given the same rate of interest.

\[
PV = \int_0^{10} f(t) e^{-0.02t} \, dt = \int_0^{10} 7000 e^{-0.03t} e^{-0.02t} \, dt
\]

\[
= \int_0^{10} 7000 e^{-0.05t} \, dt = \frac{7000}{-0.05} \left[ e^{-0.05t} \right]_0^{10}
\]

\[
= \frac{7000}{-0.05} \left( e^{-0.5} - e^0 \right) = 240,000 \left( 1 - \frac{1}{e^{0.5}} \right)
\]
(9) Find the area of the region between the curves $y = x^2 - 1$ and $y = -x - 1$.

The 2 curves intersect when $x^2 - 1 = -x - 1$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\Rightarrow x = -1, 0$

The area is

$$
\int_{-1}^{0} (-x - 1) - (x^2 - 1) \, dx = \int_{-1}^{0} (-x^2 - x) \, dx
$$

$$
= \left[ -\frac{x^3}{3} - \frac{x^2}{2} \right]_{-1}^{0} = 0 - \left( -\frac{1}{3} - \frac{1}{2} \right)
$$

$$
= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
$$
A company’s profit from selling pens and pencils is

\[ P(x, y) = 3x + 2y - .01x^2 - .02y^2 - .003xy \]

where \( x \) is the number of pens and \( y \) is the number of pencils. If 20 pens and 10 pencils have been sold what is the approximate amount of profit from selling the 11th pencil?

We need to evaluate \( \frac{\partial P}{\partial y} \) when

\( x = 20 \) & \( y = 20 \).

\[ \frac{\partial P}{\partial y} = 2 - .04y - .003x \]

\[ \frac{\partial P}{\partial y} (20, 10) = 2 - .04(10) - .003(20) \]

\[ = 2 - .4 - .6 \]

\[ = 1.54 \]
(11) The marginal cost function for an item is $MC(x) = 100 + 3x$ and the marginal revenue function is $MR(x) = 300 - x$. The initial cost is $\$50$. Find the maximum profit or minimal loss.

The marginal profit is

$$\overline{MP}(x) = MR(x) - MC(x)$$

$$= 300 - x - (100 + 3x)$$

$$= 200 - 4x.$$ 

Therefore profit is

$$P(x) = \int (200 - 4x)dx = 200x - 2x^2 + K$$

To find $K$ we obtain that

$P(0) = 0 \& C(0) = 50$ so

$$P(0) = K = 1(0) - C(0) = -50.$$ 

So $P(x) = 200x - 2x^2 - 50$.

$$\overline{MP}(x) = 0 = 200 - 4x \Rightarrow x = 50.$$ 

This critical value is the max

so the max profit is

$$P(50) = 200(50) - 2(50)^2 - 50.$$
(12) The profit function for an item is \( P(x) = 9x - x^3 - 3x^2 - 2 \) and you are only allowed to make up to 10 items. Find the maximum profit or minimal loss.

The marginal profit is

\[
\overline{P}(x) = 9 - 3x^2 - 6x = 0
\]

\Rightarrow -3(x^2 + 2x - 3) = 0

\Rightarrow -3(x + 3)(x - 1) = 0

\Rightarrow x = -3, 1.

Therefore the max occurs at

\( x = 0, 1 \) or 10.

\( P(0) = 9 - 1 - 3 - 2 = 3 \)

\( P(1) = 9 - 1 - 3 - 2 = 3 \)

\( P(10) = 9(10) - 1000 - 300 - 2 = -1208 \)

so the max is 3.
(13) The profit function for an item is \( P(x) = x + \frac{100}{x-20} + 2 \) and you are only allowed to make up to 15 items. Find the maximum profit or minimal loss.

\[
\hat{MP}(x) = 1 + \frac{-100}{(x-20)^2} = 0
\]

\[
\Rightarrow (x-20)^2 = 100 \Rightarrow x-20=10 \Rightarrow x=30
\]

So \( x=20 \) is the only CV between 0 & 15 and the max profit occurs at \( x=0, 10 \) or 15.

\[
P(0) = 0 + \frac{100}{-20} + 2 = -3
\]

\[
P(10) = 10 + \frac{100}{-10} + 2 = 2
\]

\[
P(15) = 15 + \frac{100}{15-20} + 2 = 15 - 20 + 2 = -3
\]

So the max is 2.
The demand function for a certain item is given by

\[(p + 1)(q + 6)^2 = 1000.\]

Find the elasticity of demand when \(p = 9\). If the price is increased will revenue increase or decrease?

When \(p = 9\) we have

\[(q + 1)(q + 6)^2 = 1000 \Rightarrow (q + 6)^2 = 100 \Rightarrow q + 6 = \pm 10 \Rightarrow q = 4, -16\]

Differentially

\[(1 + 6)(q + 6)^2 + (1 + 1)2(q + 6) \frac{dq}{dp} = 0.\]

When \(p = 9 \Rightarrow 2 = 4 \Rightarrow \)

\[10^2 + 60(2)1(60) \frac{dq}{dp} = 0 \Rightarrow \frac{dq}{dp} = -\frac{1}{2}
\]

\[\therefore m = -\frac{1}{4} \left(-\frac{1}{2}\right) = \frac{a}{8} > 1\]

The quantity is elastic so an increase in price will decrease the revenue.
(15) In 1988 the Lorenz curve for income distribution in the U.S. was $y = x^{2.3521}$. What was the Gini coefficient? If the Gini Coefficient in another country that some year was .4 was the income distribution more equal in this country or in the U.S.

\[
\int_0^1 (x - x^{2.3521}) \, dx = \frac{x^2}{2} - \frac{x^{3.3521}}{3.3521} \bigg|_0^1 = \frac{1}{2} - \frac{1}{3.3521}
\]

\[
G.C = 2\left(\frac{1}{2} - \frac{1}{3.3521}\right) = \frac{3.3521 - 2}{3.3521} = \frac{1.3521}{3.3521} \approx .403
\]

The distribution is more equal in the other country.
(16) Find the consumer’s surplus at the equilibrium price given that the demand function is \( p = -x^2 - 6x + 75 \) and the supply function is \( p = 2x + 10 \).

Find market equilibrium:

\[-x^2 - 6x + 75 = 2x + 10\]

\[x^2 + 8x - 65 = 0\]

\[= \) \((x + 13)(x - 5) = 0\]

\[x = -13, 5\]

Equilibrium \( x = 5, \) \( p = 20 \)

\[
\int_0^5 ((-x^2 - 6x + 75) - 20) \, dx = \int_0^5 (-x^2 - 6x + 55) \, dx
\]

\[= \left[-\frac{x^3}{3} - 3x^2 + 55x\right]_0^5 \]

\[= -\frac{125}{3} - 75 + 275 = \frac{475}{3}\]
(17) Find the consumer’s surplus at the equilibrium price given that the demand function is \( p = -x + 170 \) and the supply function is \( p = x^2 + 4x + 20 \).

\[
-x + 170 = x^2 + 4x + 20
\]

\[
\Rightarrow x^2 + 5x - 150 = 0
\]

\[
\Rightarrow (x + 15)(x - 10) = 0
\]

\[
x = 10, -15
\]

\[
p = -10 + 170 = 160
\]

\[
\int_{0}^{10} ((-x + 170) - 160) \, dx = \int_{0}^{10} (x + 10) \, dx
\]

\[
= \left[ -\frac{x^2}{2} + 10x \right]_{0}^{10}
\]

\[
C.S. = -\frac{100}{2} + 100 = 50
\]