- **Product rule:** If $F(x) = f(x)g(x)$ then

$$F'(x) = f'(x)g(x) + f(x)g'(x).$$

- **Quotient rule:** If $F(x) = \frac{f(x)}{g(x)}$ then

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

- **Chain rule:** If $F(x) = f(g(x))$ then

$$F'(x) = f'(g(x))g'(x).$$

- **Power rule:** If $F(x) = f(x)^n$ then

$$F'(x) = nf(x)^{n-1}f'(x).$$

- **Quadratic formula:** If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- **Integrals:**

$$\int f'(x)f(x)^n dx = \frac{f(x)^{n+1}}{n+1} + C \text{ if } n \neq -1;$$

$$\int f'(x)f(x)^{-1} dx = \ln|f(x)| + C;$$

$$\int f'(x)e^{f(x)} dx = e^{f(x)} + C.$$

- **Elasticity:** If q is the quantity and p is the price then the elasticity is

$$-\frac{p}{q}\frac{dq}{dp}.$$

- **Present value:** If $f(t)$ is a revenue stream and r the interest then the future value of the stream after T years is

$$\int_0^T f(t)e^{-rt} dt.$$

(1) Find the derivative $\frac{dy}{dx}$ if

(a) $y = xe^{x^2}$ $= f(x)\,g(x)$

$f(x) = x$ $\quad$ $f'(x) = 1$

$g(x) = e^{x^2}$ $\quad$ $g(x) = h(r(x))$ $\quad$ $g'(x) = h'(r(x))\,r'(x)$

$h(x) = e^{x}$ $\quad$ $h'(x) = e^{x}$

$r(x) = x^2$ $\quad$ $r'(x) = 2x$

$g'(x) = 2xe^{x^2}$

$$\frac{dy}{dx} = 1 \cdot e^{x^2} + x\,(2xe^{x^2})$$

(b) $y = \ln(x^2 + 3)$ $= f(g(x))$

$f(x) = \ln x$ $\qquad f'(x) = \frac{1}{x}$

$g(x) = x^2 + 3$ $\qquad g'(x) = 2x$

$$\frac{dy}{dx} = \frac{1}{x^2+3}(2x)$$

(c) $y = \frac{1}{\sqrt{x^2+1}}$ $= (x^2+1)^{-1/2} = f(x)^n$

$f(x) = x^2+1$

$f'(x) = 2x$

$n = -1/2$

$$\frac{dy}{dx} = -\frac{1}{2}(x^2+1)^{-\frac{1}{2}-1}(2x)$$

$$= \frac{-x}{(x^2+1)^{3/2}}$$

(d) $y = \dfrac{e^{\sqrt{x}}}{1+x} = \dfrac{f(x)}{g(x)}$

$$f(x) = h(r(x))$$

$$h(x) = e^x \qquad h'(x) = e^x$$

$$r(x) = \sqrt{x} = x^{1/2} \qquad r'(x) = \frac{1}{2}x^{-1/2}$$

$$f'(x) = e^{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$g(x) = 1+x \qquad g'(x) = 1$$

$$\frac{dy}{dx} = \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}(1+x) - e^{\sqrt{x}}\cdot 1}{(1+x)^2}$$

(e) $y = \ln(x(x^2 + 1))$ $= \ln x + \ln(x^2+1)$

$= \ln x + f(g(x))$

$g(x) = x^2+1 \qquad g'(x) = 2x$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$F'(x) = \frac{1}{x^2+1}(2x)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{2x}{x^2+1}$

$\frac{x^2+1+2x}{x(x^2+1)}$

(f) $y = e^{3x^2+1}$ $= f(g(x))$

$f(x) = e^x \qquad f'(x) = e^x$

$g(x) = 3x^2+1 \qquad g'(x) = 6x$

$$\frac{dy}{dx} = e^{3x^2+1}(6x)$$

(2) Find the second derivative of $f(x) = \frac{x}{3x+1}$. $= \dfrac{u(x)}{v(x)}$

$u(x) = x \qquad u'(x) = 1$

$v(x) = 3x+1 \qquad v'(x) = 3$

$$f'(x) = \frac{1 \cdot (3x+1) - x(3)}{(3x+1)^2}$$

$$= \frac{1}{(3x+1)^2} = (3x+1)^{-2} = g(x)^{-2}$$

$g(x) = 3x+1 \qquad g'(x) = 3$

$$f''(x) = -2(3x+1)^{-3}(3)$$

$$= \frac{-6}{(3x+1)^3}$$

(3) Find the equation of the tangent line to $y^3 = x^2 - 3$ at $(x, y) = (-2, 1)$.

$$3y^2 \frac{dy}{dx} = 2x$$

At $(x,y) = (-2, 1)$

$$3(1)\frac{dy}{dx} = 2(-2) \Rightarrow \frac{dy}{dx} = -\frac{4}{3}$$

point - slope formula:

$$y - 1 = -\frac{4}{3}(x - (-2))$$

(4) Let $f(x) = \frac{4x^2+1}{3x^2+1}$. The first derivative of f is $f'(x) = \frac{2x}{(3x^2+1)^2}$ and the second derivative is $f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$.

$$f(x) = \frac{4x^2+1}{3x^2+1}$$

$$f'(x) = \frac{2x}{(3x^2+1)^2}$$

$$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$$

(a) Find the critical values of f.

$$f(x) = \frac{4x^2+1}{3x^2+1}$$

$$f'(x) = \frac{2x}{(3x^2+1)^2}$$

$$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$$

$f'(x) = 0 \iff 2x = 0$

$\iff x = 0$

$f'(x)$ is undefined $\iff (3x^2+1)^2 = 0$

$\iff f'(x)$ is always defined

$$\boxed{\text{C.V.} \quad x = 0}$$

$f(x) = \frac{4x^2+1}{3x^2+1}$

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

$f'(x) = \frac{2x}{(3x^2+1)^2}$

$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$

The only critical value is $x=0$

So the intervals are $(-\infty,0)$ & $(0,\infty)$. We need to decide on which of these intervals the function is increasing or decreasing.

As $-1 \in (-\infty,0)$ & $f'(-1) = \frac{-2}{4^2} < 0$

f is decreasing on $(-\infty,0)$

As $1 \in (0,\infty)$ & $f'(1) = \frac{2}{4^2} > 0$

f is increasing on $(0,\infty)$.

$f(x) = \frac{4x^2+}{3x^2+1}$

$f'(x) = \frac{2x}{(3x^2+1)^2}$

$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$

(c) Find the intervals where f is concave up and the intervals where f is concave down.

We first find where $f''(x) = 0$.

This occur when $2(1-9x^2) = 0$

$\iff 9x^2 = 1 \iff x = \pm\frac{1}{3}$.

So our intervals are $(-\infty, -\frac{1}{3})$, $(-\frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{3}, +\infty)$. Choosing pts. in each interval we see that $f''(-1) = \frac{2(-8)}{4^3} < 0$

$f''(0) = \frac{2}{1^3} > 0$ & $f''(1) = \frac{2(-8)}{4^3} < 0$

We have f is concave up on $(-\frac{1}{3}, \frac{1}{3})$ & concave down on

$(-\infty, -\frac{1}{3})$ & $(\frac{1}{3}, +\infty)$.

$f(x) = \frac{4x^2+1}{3x^2+1}$

$f'(x) = \frac{2x}{(3x^2+1)^2}$

$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$

(d) Find x-values for any inflection points for f.

The inflection pts are where concavity changes so they are $x = \pm \frac{1}{3}$.

(e) Find the x-values of any relative maxima and minima. Make sure you clear which values correspond to maxima and which to minima.

$$f(x) = \frac{4x^2+1}{3x^2+1}$$

$$f'(x) = \frac{2x}{(3x^2+1)^2}$$

$$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$$

The relative extrema occur at the critical values. Here $x=0$ is the only critical value. We need to determine if it is a relative max or min.

Note that $f''(0) > 0$ so by the 2nd derivative test $x=0$ is a relative min.

(f) Find the horizontal and vertical asymptotes.

$f(x) = \frac{4x^2+1}{3x^2+1}$

$f'(x) = \frac{2x}{(3x^2+1)^2}$

$f''(x) = \frac{2(1-9x^2)}{(3x^2+1)^3}$

To find the horizontal asymptote we take $\lim_{x \to \infty} \frac{4x^2+1}{3x^2+1} = \lim_{x \to \infty} \frac{4x^2}{3x^2} = \frac{4}{3}$.

So the horizontal asymptote is $y = \frac{4}{3}$.

Vertical asymptotes occur when the denominator of $f(x)$ is zero. Since $3x^2+1 > 1$ for all values of x there are no vertical asymptotes.

(5) If $f(x,y) = \dfrac{x}{y-x}$, determine the following:

$f(x,y) = \dfrac{u(x,y)}{v(x,y)}$ where

$u(x,y) = x$ and $v(x,y) = y-x$.

Taking partial derivatives we have

$\dfrac{\partial u}{\partial x} = 1$ & $\dfrac{\partial v}{\partial x} = -1$ so

$$\frac{\partial f}{\partial x} = \frac{\frac{\partial u}{\partial x}\cdot v - u\cdot\frac{\partial v}{\partial x}}{v^2}$$

$$= \frac{1\cdot(y-x) - x\cdot(-1)}{(y-x)^2}$$

$$= \frac{y}{(y-x)^2}$$

(a) $\dfrac{\partial f}{\partial x}$

(b) $\frac{\partial^2 f}{\partial x \partial y}$

As $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ we need to take the partial derivative of $\frac{\partial f}{\partial x} = \frac{y}{(y-x)^2}$ with respect to y.

We have $\frac{y}{(y-x)^2} = \frac{u(x,y)}{v(x,y)}$ where $u(x,y) = y$ & $v(x,y) = (y-x)^2$ so

$\frac{\partial u}{\partial y} = 1$ & $\frac{\partial v}{\partial y} = 2(y-x)(1)$.

Therefore

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1 \cdot (y-x)^2 - y(2(y-x))}{(y-x)^4}$$

(6) Compute the following integrals:

We use the log rule with

$u = x^2+1$ & $u' = 2x$

Since $\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\int \frac{2x}{x^2+1}\,dx$

we have

$$\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\left(\ln|x^2+1| + C\right).$$

(a) $\int \frac{3x}{x^2+1}\,dx$

(b) $\int \frac{x}{(x^2+2)^2}\,dx = \int x\,(x^2+2)^{-2}\,dx.$

Let $u = x^2+2$ & $n=-2$. So $u' = 2x$. Then by the power rule

$$\int 2x\,(x^2+2)^{-2}\,dx = \frac{(x^2+2)^{-1}}{-1} + C$$

& since

$$\int \frac{x}{(x^2+2)^2}\,dx = \frac{1}{2}\int 2x\,(x^2+2)^{-2}\,dx$$

we have

$$\int \frac{x}{(x^2+2)^2}\,dx = \frac{1}{2}\left(\frac{(x^2+2)^{-1}}{-1} + C\right)$$

(c) $\int 8x\sqrt{x^2-1}\,dx \quad = \int 8x\,(x^2-1)^{\frac{1}{2}}\,dx$

$$4\int 2x\,(x^2-1)^{1/2}\,dx$$

$$= 4\left(\frac{(x^2-1)^{\frac{3}{2}}}{3/2} + C\right)$$

Let $u = x^2-1$ so $u' = 2x$

& $n = \frac{1}{2}$. To apply the power rule

we see that . For the red integral

to equal the black we need to multiply

by 4. Therefore

$$\int 8x\sqrt{x^2-1}\,dx = 4\left(\frac{(x^2-1)^{3/2}}{3/2} + C\right).$$

(d) $\int_0^1 \frac{x}{e^{x^2}}\,dx = \int_0^1 x\, e^{-x^2}\,dx$

$u = -x^2 \qquad u' = -2x$

so $\int_0^1 \frac{x}{e^{x^2}}\,dx = -\frac{1}{2}\int_0^1 -2x\; e^{-x^2}\,dx$

$= -\frac{1}{2}\left(e^{-x^2} \Big|_0^1 \right) = -\frac{1}{2}\left(e^{-1} - e^0\right)$

$= \frac{1}{2}\left(1 - \frac{1}{e}\right).$

(e) $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx \;=\; \int_0^4 x\,(x^2+9)^{-1/2}\,dx$

$u = x^2+9$ and $u' = 2x$ so

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int_0^4 2x\,(x^2 \quad ^{-1}\,dx$$

$$= \frac{1}{2}\left(\frac{(x^2+9)^{1/2}}{1/2} \Big|_0^4 \right.$$

$$= \frac{1}{2}\left(\frac{\sqrt{16+9}}{1/2} - \frac{\sqrt{9}}{1/2} \right) = 5 - 3 = 2$$

(7) Suppose that a donor wishes to provide a cash gift to the University that will generate a continuous income stream with an annual rate of flow at time t given by $f(t) = \$80,000$ per year. If the annual interest rate of 4% compounded continuously, find the capital value of this income stream.

Find the present value of the income stream after 10 years.

$$PV = \int_0^{10} f(t)\, e^{-.04t}\, dt$$

$$= \int_0^{10} 8000\, e^{-.04t}\, dt$$

$$= \frac{8000}{-.04} \int_0^{10} (-.04)\, e^{-.04t}\, dt$$

$$= \frac{8000}{-.04} \left(e^{-.04t} \Big|_0^{10} \right)$$

$$= \frac{8000}{-.04} \left(e^{-.4} - e^{0} \right)$$

$$= 200,000 \left(1 - \frac{1}{e^{.4}} \right).$$

(8) A company has an income stream of $f(t) = 7000e^{-.03t}$ in dollars per year. Find the present value of the income stream over the next 10 years if it is invested at a rate of 2% a year compounded continuously. Also find the capital value of the income stream given the same rate of interest.

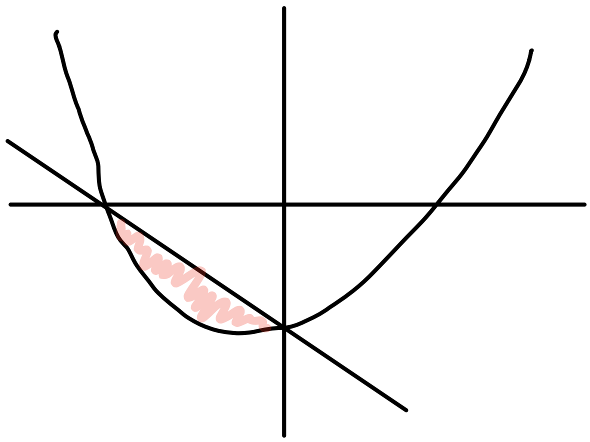
$$PV = \int_0^{10} f(t)\, e^{-.02t}\, dt = \int_0^{10} 7000\, e^{-.03t}\, e^{-.02t}\, dt$$

$$= \int_0^{10} 7000\, e^{-.05t}\, dt = \frac{7000}{-.05} \int_0^{10} -.05\, e^{-.05t}\, dt$$

$$= \frac{7000}{-.05} \left(e^{-.05t} \Big|_0^{10} \right)$$

$$= -240{,}000 \left(e^{-.5} - e^{0} \right) = 240{,}000 \left(1 - \frac{1}{e^{.5}} \right)$$

(9) Find the area of the region between the curves $y = x^2 - 1$ and $y = -x - 1$.



The 2 curves intersect when $x^2 - 1 = -x - 1$

$\Rightarrow \quad x^2 + x = 0$

$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = -1, 0$

The area is

$$\int_{-1}^{0} \left((-x-1) - (x^2-1) \right) dx = \int_{-1}^{0} (-x^2 - x)\, dx$$

$$= \left(-\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_{-1}^{0} = 0 - \left(-\frac{-1}{3} - \frac{1}{2} \right)$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(10) A company's profit from selling pens and pencils is

$$P(x, y) = 3x + 2y - .01x^2 - .02y^2 - .003xy$$

where x is the number of pens and y is the number of pencils. If 20 pens and 10 pencils have been sold what is the approximate amount of profit from selling the 11th pencil?

We need to evaluate $\frac{\partial P}{\partial y}$ when

$x = 20$ & $y = 10$.

$$\frac{\partial P}{\partial y} = 2 - .04y - .003x$$

$$\frac{\partial P}{\partial y}(20, 10) = 2 - .04(10) - .003(20)$$

$$= 2 - .4 - .06$$

$$= 1.54$$

(11) The marginal cost function for an item is $\overline{MC}(x) = 100 + 3x$ and the marginal revenue function is $\overline{MR}(x) = 300 - x$. The initial cost is \$50. Find the maximum profit or minimal loss.

The marginal profit is

$$\overline{MP}(x) = \overline{MR}(x) - \overline{MC}(x)$$
$$= 300 - x - (100 + 3x)$$
$$= 200 - 4x.$$

Therefore profit is

$$P(x) = \int (200 - 4x)\,dx = 200x - 2x^2 + K$$

To find K we observe that

$R(0) = 0$ & $C(0) = 50$ so

$$P(0) = K = R(0) - C(0) = -50.$$

So $P(x) = 200x - 2x^2 - 50.$

$\overline{MP}(x) = 0 = 200 - 4x \Rightarrow x = 50.$

This critical value is the max

so the max profit is

$$P(50) = 200(50) - 2(50)^2 - 50.$$

(12) The profit function for an item is $P(x) = 9x - x^3 - 3x^2 - 2$ and you are only allowed to make up to 10 items. Find the maximum profit or minimal loss.

The marginal profit is

$\overline{MP}(x) = 9 - 3x^2 - 6x = 0$

$\Rightarrow -3(x^2 + 2x - 3) = 0$

$\Rightarrow -3(x+3)(x-1) = 0$

$\Rightarrow x = -3, 1.$

Therefore the max occurs at

$x = 0, 1$ or 10.

$P(0) = -2$

$P(1) = 9 - 1 - 3 - 2 = 3$

$P(10) = 9(10) - 1000 - 300 - 2 = -1208$

So the max is 3.

(13) The profit function for an item is $P(x) = x + \dfrac{100}{x-20} + 2$ and you are only allowed to make up to 15 items. Find the maximum profit or minimal loss.

$$\overline{MP}(x) = 1 + \frac{-100}{(x-20)^2} = 0$$

$$\Rightarrow (x-20)^2 = 100 \quad \Rightarrow \quad x-20 = \pm 10$$

$$\Rightarrow x = 10, 30$$

So $x=10$ is the only C.V between 0 & 15 and the max profit occurs at $x=0, 10$ or 15.

$$P(0) = 0 + \frac{100}{-20} + 2 = -3$$

$$P(10) = 10 + \frac{100}{-10} + 2 = 2$$

$$P(15) = 15 + \frac{100}{15-20} + 2 = 15 - 20 + 2 = -3$$

So the max is 2.

(14) The demand function for a certain item is given by

$$(p+1)(q+6)^2 = 1000\,.$$

Find the elasticity of demand when $p = 9$. If the price is increased will revenue increase or decrease?

When $p=9$ we have

$$(9+1)(q+6)^2 = 1000 \Rightarrow (q+6)^2 = 100$$

$$\Rightarrow q+6 = \pm 10$$

$$\Rightarrow q = 4, -16$$

Differentiating

$$(1+0)(q+6)^2 + (p+1)\,2(q+6)\frac{dq}{dp} = 0\,.$$

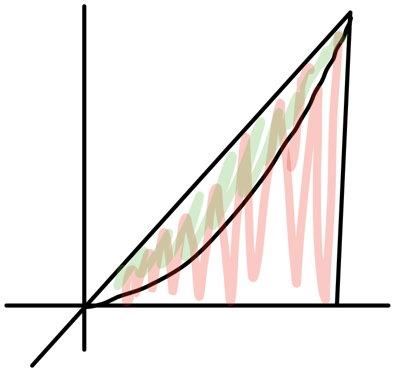
When $p=9$ & $q=4$ $\Rightarrow$

$$10^2 + 10(2)(10)\frac{dq}{dp} = 0 \Rightarrow \frac{dq}{dp} = -\frac{1}{2}$$

& $\eta = -\frac{9}{4}\left(-\frac{1}{2}\right) = \frac{9}{8} > 1$

The quantity is elastic so an increase in price will decrease the revenue.

(15) In 1988 the Lorenz curve for income distribution in the U.S. was $y = x^{2.3521}$. What was the Gini coefficient? If the Gini Coefficient in another country that some year was .4 was the income distribution more equal in this country or in the U.S.



$$G.C = \frac{\text{Area}}{\text{Area} = \frac{1}{2}}$$

$$= 2(\text{Area}).$$

$$\int_0^1 (x - x^{2.3521})\,dx$$

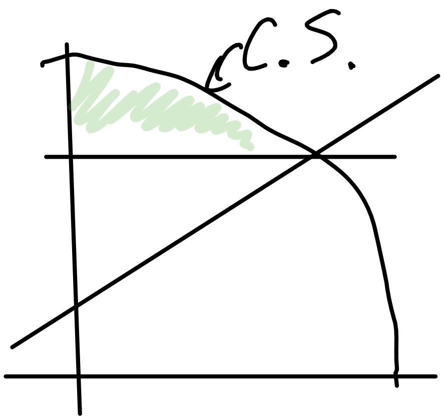
$$= \left.\frac{x^2}{2} - \frac{x^{3.3521}}{3.3521}\right|_0^1 = \frac{1}{2} - \frac{1}{3.3521}$$

$$G.C = 2\left(\frac{1}{2} - \frac{1}{3.3521}\right) = \frac{3.3521 - 2}{3.521}$$

$$= \frac{1.3521}{3.3521} \approx .403$$

The distribution is more equal in the other country.

(16) Find the consumer's surplus at the equilibrium price given that the demand function is $p = -x^2 - 6x + 75$ and the supply function is $p = 2x + 10$.



Find market equilibrium:

$$-x^2 - 6x + 75 = 2x + 10$$

$$x^2 + 8x - 65 = 0$$

$$\Rightarrow (x+13)(x-5) = 0$$

$$x = -13, 5$$

$$p = 2(5) + 10 = 20$$

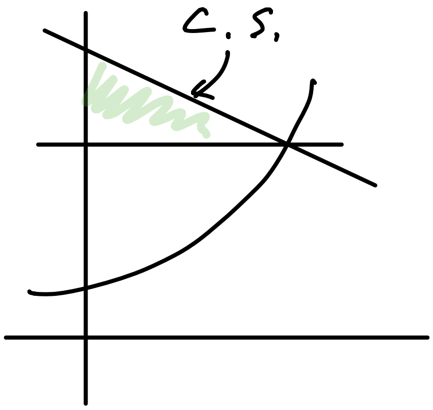
Equilibrium $x = 5$, $p = 20$

$$\int_0^5 \left((-x^2 - 6x + 75) - 20 \right) dx = \int_0^5 (-x^2 - 6x + 55)\, dx$$

$$= -\frac{x^3}{3} - 3x^2 + 55x \Big|_0^5$$

$$= -\frac{125}{3} - 75 + 275 = \frac{475}{3}$$

(17) Find the consumer's surplus at the equilibrium price given that the demand function is $p = -x + 170$ and the supply function is $p = x^2 + 4x + 20$.



$$-x + 170 = x^2 + 4x + 20$$

$$\Rightarrow x^2 + 5x - 150 = 0$$

$$\Rightarrow (x+15)(x-10) = 0$$

$$x = 10, -15$$

$$p = -10 + 170 = 160$$

$$\int_0^{10} \left((-x+170) - 160 \right) dx = \int_0^{10} (-x + 10)\, dx$$

$$= \left(-\frac{x^2}{2} + 10x \right) \Big|_0^{10}$$

$$CS = -\frac{100}{2} + 100 = 50$$