In a previous homework we constructed Cantor sets in $[0,1]$ with measure any $\epsilon < 1$. A closer look at the construction shows that for any open interval I and any $\epsilon < 1$ we can find a Cantor set $C \subset I$ with the following properties:

(A) $m(C) = \epsilon m(I)$;

(B) C is closed and nowhere dense with no isolated points.

(C) Each component of $I \backslash C$ is an interval of width $< m(I)/2$.

We use Cantor sets to construct the set E.

Lemma 0.1 *For any $\epsilon > 0$ there exist Borel sets $C_n \subset [0,1]$ with the following properties:*

(1) $C_n \subset C_{n+1}$;

(2) C_n is closed and nowhere dense;

(3) each component of $(0,1)\backslash C_n$ is an interval of measure less than $1/2^n$;

(4) if I is a component of $(0,1)\backslash C_{n-1}$ then $I \cap C_n = \emptyset$ if $m(I) < 1/2^n$ and $0 < m(I \cap C_n) < \epsilon/2^{3n+2}$ otherwise.

Proof. We proceed by induction. Let $C_0 = \emptyset$.

Assuming the sets $C_0, \ldots, C_{n_1}$ have been constructed we build C_n. The complement $[0,1]\backslash C_{n-1}$ can contain at most 2^n intervals of measure $> 1/2^n$. Label them $I_1, \ldots, I_k$. Let D_i be a Cantor set in I_i with $m(D_i) = \epsilon/2^{3n+2}$ and such that each component of $I_i \backslash D_i$ has measure $< m(I_i)/2 < 1/2^n$ where the last inequality follows from $m(I_i) < 1/2^{n-1}$.. Let $A_n = \cup_{i=1}^k D_i$ and $C_n = A_n \cup C_{n-1}$. We need to show that C_n satisfies (1)-(4).

Property (1) clearly holds. For (2) we note that a finite union of closed, nowhere dense sets is closed and nowhere dense so A_n and $C_n = C_{n-1} \cup A_n$ are closed and nowhere dense.

Every component I of $[0,1]\backslash C_n$ is contained in some component I' of $[0,1]\backslash C_{n-1}$. If $m(I') < 1/2^n$ then $m(I) < 1/2^n$ (and $I' = I$). Otherwise I is a component of $I' \backslash D'$ where D' is a Cantor set, as above, and again $m(I) < 1/2^n$. This gives (3).

Property (4) follows directly from our construction. 0.1

Lemma 0.2 *Let $E = \cup C_n$ where the C_n are sets from Lemma 0.1. If $I \subset (0,1)$ is a component of $(0,1)\backslash C_n$ then $0 < m(I \cap E) < \epsilon m(I)$.*

Proof. By (3), $m(I) < 1/2^n$. We will bound $m(I \cap (C_k \backslash C_{k-1}))$ for $k \geq n$. We first note that there exists an $m \geq n$ such that $1/2^{m+1} \leq m(I) < 1/2^m$. We then have:

- If $k < m$, then by (4), $I \cap C_{k+1} = \emptyset$ so $m(I \cap (C_{k+1} \backslash C_k)) = 0$.
- If $k = m$, then (4) implies that $0 < m(I \cap C_{k+1}) < \epsilon/2^{3(k+1)+2}$. Since $I \cap C_k = \emptyset$ we have $m(I \cap (C_{k+1} \backslash C_k)) = m(I \cap C_k)$.
- If $k > m$, then every component of $(0,1)\backslash C_k$ that intersects I will be contained in I. In particular there are at most $\lfloor m(I)/(1/2^{k+1}) \rfloor \leq 2^{k+1-m}$ components of $(0,1)\backslash C_k$ of measure $\geq 1/2^{k+1}$, that intersect I. Property (4) then implies that $m(I \cap (C_{k+1}/C_k)) \leq 2^{k+1-m}\epsilon/2^{3(k+1)+2} = \epsilon/2^{2k-m+4}$.

We then calculate

$$\begin{aligned} m(I \cap E) &= \sum_{k=1}^{\infty} m(I \cap (C_k \backslash C_{k-1})) \\ &= m(I \cap (C_m \backslash C_{m-1})) + \sum_{k=m+1}^{\infty} m(I \cap (C_k \backslash C_{k-1})) \\ &\leq \frac{\epsilon}{2^{3(m+1)+2}} + \sum_{k=m+1}^{\infty} \frac{\epsilon}{2^{2k-m+4}} \\ &= \frac{\epsilon}{2^{3(m+1)+2}} + \frac{\epsilon}{2^{m+2}} \\ &< \frac{\epsilon}{2^{m+1}} \end{aligned}$$

and therefore $m(I \cap E) < \epsilon m(I)$.

Note that the second bullet implies that $m(I \cap E) > 0$. 0.2

Note that if we apply the previous lemma to $(0,1) = (0,1)\backslash C_0$ we see that $m(I \cap E) = m(E) < \epsilon$.

Lemma 0.3 *If $J \subset (0,1)$ is an interval then $0 < m(E \cap J) < m(J)$.*

Proof. We need to show that for some n there is a component I of $(0,1)\backslash C_n$ such that $I \subset J$. Then by Lemma 0.2, $m(J \cap E) \geq m(I \cap E) > 0$ and $m(J \cap E) = m((J \backslash I) \cap E) + m(I \cap E) < m(J \backslash I) + m(I) = m(J)$.

Pick an x in the interior of I. Then there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset I$. Fix $n > 0$ such that $1/2^{n-1} < \delta$. Since C_n is nowhere dense there exists a $y \in (0,1)\backslash C_n$ such that $|x - y| < 1/2^n$. By property (3) y is a contained in a component I of $(0,1)\backslash C_n$ with $m(I) < 1/2^n$ and therefore $I \subset (x - \delta, x + \delta) \subset J$. 0.3

We have now constructed the desired set on the interval $(0,1)$. By translating a copy of the set E to each interval $(n, n+1)$ we get such a set on $\mathbb{R}$. Note that we can choose each translate to have measure $1/(1 + |n|^2)$ so that the total set has finite measure.