From Rudin: Chapter 1, # 3,5,6

1. Show that every Riemann integrable function is measurable (with respect to the \( \sigma \)-algebra \( M \) on which Lesbesgue measure is defined). Show that the Riemann integral and Lesbesgue integral have the same value. You can just do the case for a 1 variable functions on \([0,1]\). Here is an outline of a proof.

   (a) Show that there exists a measurable function \( g \) with \( f = g \) a.e.

   (b) Recall that we defined a ”measure” for any subset of \( \mathbb{R}^n \) but this ”measure” was only countably additive on a certain \( \sigma \)-algebra \( M \). Show that any set with measure zero is in \( M \).

   (c) Let \( f_0 \) and \( f_1 \) be functions such that \( f_0 = f_1 \) a.e. Show that if \( f_0 \) is measurable then \( f_1 \) is measurable.

   (d) Conclude that \( f \) is measurable.

2. For this problem we will define and study a notion of convergence of positive measures on \((\mathbb{R}^n, B)\) (recall that \( B \) is the \( \sigma \)-algebra of Borel sets). Let \( \mu_i \) and \( \mu \) be positive measures. Then \( \mu_i \to \mu \) if for every continuous, compactly supported function \( f : \mathbb{R}^n \to [0, \infty] \) we have

\[
\int_{\mathbb{R}^n} f \, d\mu_i \to \int_{\mathbb{R}^n} f \, d\mu.
\]

(a) Let

\[
g_i : \mathbb{R}^n \to [0, \infty]
\]

be measurable functions. As in class, for each \( g_i \) define a measure \( \phi_i \) by the formula

\[
\phi_i(E) = \int_E g_i \, dm
\]

where \( m \) is Lesbesgue measure. Let \( g \) also be a non-negative measurable function and assume that

\[
\lim_{i \to \infty} \int_{\mathbb{R}^n} |g - g_i| \, dm = 0.
\]

Let \( \phi \) be the measuring corresponding to \( g \). Show that \( \phi_i \to \phi \).

(b) Let \( \mu_k \) be the counting measures defined in class. Show that \( \mu_k \to m \). (Here is the definition of \( \mu_k \). For each non-negative integer \( k \) define

\[
\frac{1}{2^k} \mathbb{Z}^n = \left\{ \vec{x} \in \mathbb{R}^n | 2^k \vec{x} \in \mathbb{Z}^n \right\}.
\]

Then the measure \( \mu_k \) is defined by the formula

\[
\mu_k(E) = \sum_{x \in \frac{1}{2^k} \mathbb{Z}^n} \frac{1}{2^k} \mu_x(E)
\]

1
where

\[ \mu_x(E) = \begin{cases} 
1 & x \in E \\
0 & x \notin E 
\end{cases} \]

is the atomic measure with support at \( x \).