6720: Complex Analysis

Professor Paul C. Bressloff

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Recommended texts

M. J. Ablowitz and A. S. Fokas *Complex variables* (Cambridge University Press, 1997)

J. P. Keener *Principles of applied mathematics* (Perseus Press, 2000)

Complex analysis

Assignment II (Deadline March 25th)

[1]. Consider a flow around a cylindrical obstacle of radius a with the complex potential $\Omega(z) = u_0(z + a^2/z)$. Let P and P_∞ denote the pressure at a point on the cylinder and far from the cylinder, respectively.

(a) Use Bernoulli's equation $P + \rho|u|^2/2 = c$ along streamlines to show that

$$P - P_\infty = \frac{1}{2}\rho u_0^2(1 - 4\sin^2\theta)$$

(b) Show that a vacuum is created at the points $\pm ia$ if the velocity of the fluid satisfies $u_0 \geq \sqrt{2P_\infty/(3\rho)}$. This phenomenon is often called *cavitation*.

[2]. Find a conformal map f that maps the region between tweo circles $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ onto an annulus $\rho_0 < |z| < 1$ and find ρ_0.

[3]. Determine the conformal map f that takes the region D exterior to the pair of circular discs of radius R to the region outside a circular disc of unit radius centered at the origin (see figure 1). Use the following steps:

(a) Show how the map $\omega = 2R/z$ takes the region D to the infinite strip $-i < \mathrm{Im}\,\omega < i$.

(b) Show how the map $\xi = e^{\pi(i-\omega)/2}$ takes the infinite strip to the upper-half complex ξ-plane

(c) Show how the map $\eta = (\xi + i)/(\xi - i)$ takes the upper-half complex ξ-plane to the region outside the unit disc, that is, $|\eta| > 1$.

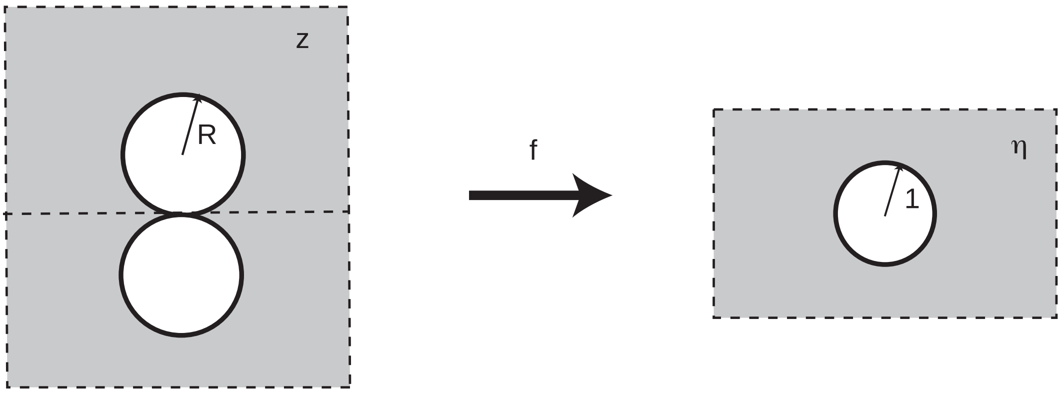

Figure 1: Find the conformal map f

[4]. A circular cylinder of radius R lies at the bottom of a channel of fluid that at a large distance from the cylinder has a constant velocity u_0 in the horizontal direction.

(a) Show that the complex potential is given by

$$\Omega(z) = \pi R u_0 \coth\left(\frac{\pi R}{z}\right)$$

[Hint: Use the conformal map found in [3] and a symmetry argument. Also need to calcualte asymptotic velocity in η plane.]

(b) Use Bernoulli's equation to show that the difference in pressure between the top and the bottom points of the cylinder is $\rho\pi^4 u_0^2/32$ where ρ is the density of the fluid.

[5] Consider the Schwarz-Cristoffel transformation from a closed n-sided polygon (with interior angles $\pi\alpha_i$, $i = 1, ..., n$) to the upper-half complex plane defined by

$$\frac{d\omega}{dz} = \gamma \prod_{i=1}^{n} (z - a_i)^{\alpha_i - 1} \tag{1}$$

For a closed polygon $\sum_{i=1}^{n} \alpha_i = n - 2$.

(a) Using the bilinear transformation

$$z = i\frac{1+\zeta}{1-\zeta}$$

which transforms the upper half z-plane onto the unit disc $|\zeta| < 1$, show that the conformal map from a closed polygon to the unit disc is also given by the Schwarz-Cristoffel transformation with $z \to \zeta$, $\gamma \to \hat{\gamma}$ and $a_i \to \zeta_i$ where the points ζ_i lie on the unit circle.

(b) Show that the function

$$\omega = \int_0^z \frac{1}{(1-t^6)^{1/3}} dt$$

maps a regular hexagon into the unit disc. [Hint: use a symmetry argument]