Bifurcation theory: Problems III

[3.1] Find all the absolutely irreducible representations of $\mathcal{D}_3$, the symmetry group of an equilateral triangle, and hence work out all the possible solutions that are guaranteed at a steady bifurcation with $\mathcal{D}_3$ symmetry under one or other of these representations. Draw examples of the eigenmodes in an appropriate triangular box, and work out the relevant normal form equations.

[3.2] Repeat the previous exercise for $\mathcal{D}_6$, the symmetry group of a regular hexagon, this time drawing the eigenmodes in a hexagonal domain. Hint: Use the fact that there are 6 conjugacy classes: $\{e\}$, $\{\rho^3\}$, $\{\rho, \rho^5\}$, $\{\rho^2, \rho^4\}$, $\{m\rho, m\rho^3, m\rho^5\}$ (reflections with axis corner to corner) and $\{m, m\rho^2, m\rho^4\}$ (reflections with axis edge to edge).

[3.3] Analyze the Hopf bifurcation with Z_2 symmetry where the action of the group $Z_2 \times S^1$ on the amplitude z is given by

$$(e, \theta)z = \mathrm{e}^{i\theta} z, \quad (m, 0)z = -z$$

for $z \in C$. What is the spatiotemporal symmetry of the guaranteed solution nb. consider a pattern in a rectangular box given by $u(\mathbf{x}, t) = z(t)A(\mathbf{x}) + c.c.$ such that $A(m\mathbf{x}) = A(-x_1, x_2) = -A(x_1, x_2)$.

[3.4] Repeat the previous exercise for Z_4 symmetry where the action of the group $Z_4 \times S^1$ on z is

$$(e, \theta)z = \mathrm{e}^{i\theta} z, \quad (\rho, 0)z = -z$$

Here ρ corresponds to rotation in the plane by $\pi/2$. What is the spatiotemporal symmetry of the guaranteed solution in a square box.

[3.5] Work out the cubic amplitude equations for the steady bifurcation on a square lattice when the system is weakly isotropic so that x_1, x_2 directions are not equivalent nb. system is no longer equivariant with respect to ρ (rotation by $\pi/2$) but is still equivariant with respect to m, ρ^2 or ,equivalently, reflections about the horizontal and vertical axes. What happens to the square solutions?

[3.6] Work out the amplitude equation for a Hopf bifurcation on a square lattice, where the pattern takes the form

$$u(\mathbf{x}, t) = z_1 \mathrm{e}^{i(x_1 - t)} + z_2 \mathrm{e}^{-i(x_1 + t)} + z_3 \mathrm{e}^{i(x_2 - t)} + z_4 \mathrm{e}^{-i(x_2 + t)} + c.c.$$

where $z_j \in C$.