Bifurcation theory: Problems II

[2.1] Consider the following ODE from ecology, which models a single population under a constant harvest:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \alpha$$

where x is the population number, r and K are the intrinsic growth rate and the carrying capacity of the population, respectively, and α is the harvest rate, which is a control parameter. Find a parameter value α_0 at which the system exhibits a saddle–node bifurcation, and check the genericity conditions of the saddle–node bifurcation theorem. Based on this analysis, explain what might be a result of overharvesting on the ecosystem dynamics.

[2.2] Write the system

$$\dot{x} = -y - xy + 2y^2, \quad \dot{y} = x - x^2 y$$

in terms of the complex coordinate $z = x + iy$ and compute the normal form coefficient $l_1(0)$. Is the origin stable?

[2.3] Check that the Van der Pol oscillator

$$\ddot{y} - (\alpha - y^2)\dot{y} + y = 0$$

has a fixed point that exhibits the Hopf bifurcation at some value of α and compute $l_1(0)$.

[2.4] Normal form of flip bifurcation. Consider the normal form of the so–called flip bifurcation, which is given by the 1D map

$$x' = -(1 + \alpha)x + x^3 \equiv f_\alpha(x)$$

Determine the stability of the fixed point at the origin as a function of α and show that the fixed point is non hyperbolic at $\alpha = 0$. [Note: for discrete dynamical systems a fixed point is nonhyperbolic if the linearized map has an eigenvalue on the unit circle]. By considering the second iterate $f^2_\alpha(x)$ show that there exist two additional fixed points when $\alpha > 0$ and determine their stability. Sketch a bifurcation diagram.

[2.5] Generic flip bifurcation. Suppose that a one–dimensional discrete dynamical system

$$x' = f(x, \alpha), \quad x, \alpha \in \mathbf{R}$$

with smooth f has at $\alpha = 0$ the fixed point $x = 0$ and let $f_x(0, 0) = -1$. Assume that the following genericity conditions are satisfied:

$$\frac{1}{2}(f_{xx}(0,0))^2+\frac{1}{3}f_{xxx}(0,0)\neq 0 \qquad [\mathbf{G1}]$$

$$f_{x\alpha}(0,0)\neq 0 \qquad [\mathbf{G2}]$$

(a) Use the implicit function theorem to show that there exists a unique fixed point in some neighborhood of the origin for sufficiently small $|\alpha|$. Without loss of generality we can take this fixed point to be at the origin for all sufficiently small $|\alpha|$. Why?

(b) Taylor expand the map f as

$$f(x,\alpha)=f_1(\alpha)x+f_2(\alpha)x^2+f_3(\alpha)x^3+\mathcal{O}(x^4)$$

where $f_1(\alpha)=-[1+g(\alpha)]$ for some some smooth g. Using the bifurcation conditions and **[G2]** show that g is locally invertible and can thus be used to introduce a new parameter $\beta=g(\alpha)$ such that

$$x'=-(1+\beta)x+a(\beta)x^2+b(\beta)x^3+\mathcal{O}(x^4)$$

where

$$a(0)=f_2(0)=\frac{1}{2}f_{xx}(0,0), \quad b(0)=\frac{1}{6}f_{xxx}(0,0)$$

(c) By performing the change of coordinates $x=y+\delta y^2$ show that the quadratic term can be eliminated on setting

$$\delta(\beta)=\frac{a(\beta)}{(1+\beta)^2+(1+\beta)}$$

and thus

$$y'=-(1+\beta)y+c(\beta)y^3+\mathcal{O}(y^4)$$

for some smooth $c(\beta)$ such that

$$c(0)=a^2(0)+b(0)=\frac{1}{4}(f_{xx}(0,0))^2+\frac{1}{6}f_{xxx}(0,0)$$

(d) Finally, by performing the rescaling $y=\eta/\sqrt{|c(\beta)|}$ obtain the normal form (to cubic order)

$$\eta'=-(1+\beta)\eta+\sigma\eta^3$$

where $\sigma=\text{sign } c(0)=\pm 1$.