Bifurcation theory: Problems II

[2.1] Consider the following ODE from ecology, which models a single population under a constant harvest:

\[
\dot{x} = rx \left(1 - \frac{x}{K}\right) - \alpha
\]

where \(x\) is the population number, \(r\) and \(K\) are the intrinsic growth rate and the carrying capacity of the population, respectively, and \(\alpha\) is the harvest rate, which is a control parameter. Find a parameter value \(\alpha_0\) at which the system exhibits a saddle–node bifurcation, and check the genericity conditions of the saddle–node bifurcation theorem. Based on this analysis, explain what might be a result of overharvesting on the ecosystem dynamics.

[2.2] Write the system

\[
\dot{x} = -y - xy + 2y^2, \quad \dot{y} = x - x^2y
\]

in terms of the complex coordinate \(z = x + iy\) and compute the normal form coefficient \(l_1(0)\). Is the origin stable?

[2.3] Check that the Van der Pol oscillator

\[
\ddot{y} - (\alpha - y^2)\dot{y} + y = 0
\]

has a fixed point that exhibits the Hopf bifurcation at some value of \(\alpha\) and compute \(l_1(0)\).

[2.4] **Normal form of flip bifurcation.** Consider the normal form of the so–called flip bifurcation, which is given by the 1D map

\[
x' = -(1 + \alpha)x + x^3 \equiv f_\alpha(x)
\]

Determine the stability of the fixed point at the origin as a function of \(\alpha\) and show that the fixed point is non hyperbolic at \(\alpha = 0\). [Note: for discrete dynamical systems a fixed point is nonhyperbolic if the linearized map has an eigenvalue on the unit circle]. By considering the second iterate \(f^2_\alpha(x)\) show that there exist two additional fixed points when \(\alpha > 0\) and determine their stability. Sketch a bifurcation diagram.

[2.5] **Generic flip bifurcation.** Suppose that a one–dimensional discrete dynamical system

\[
x' = f(x, \alpha), \quad x, \alpha \in \mathbb{R}
\]

with smooth \(f\) has at \(\alpha = 0\) the fixed point \(x = 0\) and let \(f_x(0, 0) = -1\). Assume that the following genericity conditions are satisfied:
\[
\frac{1}{2} (f_{xx}(0, 0))^2 + \frac{1}{3} f_{xxx}(0, 0) \neq 0 \quad [G1]
\]
\[
f_{x\alpha}(0, 0) \neq 0 \quad [G2]
\]

(a) Use the implicit function theorem to show that there exists a unique fixed point in some neighborhood of the origin for sufficiently small \(|\alpha|\). Without loss of generality we can take this fixed point to be at the origin for all sufficiently small \(|\alpha|\). Why?

(b) Taylor expand the map \(f\) as
\[
f(x, \alpha) = f_1(\alpha)x + f_2(\alpha)x^2 + f_3(\alpha)x^3 + O(x^4)
\]
where \(f_1(\alpha) = -[1 + g(\alpha)]\) for some some smooth \(g\). Using the bifurcation conditions and \([G2]\) show that \(g\) is locally invertible and can thus be used to introduce a new parameter \(\beta = g(\alpha)\) such that
\[
x' = -(1 + \beta)x + a(\beta)x^2 + b(\beta)x^3 + O(x^4)
\]
where
\[
a(0) = f_2(0) = \frac{1}{2} f_{xx}(0, 0), \quad b(0) = \frac{1}{6} f_{xxx}(0, 0)
\]

(c) By performing the change of coordinates \(x = y + \delta y^2\) show that the quadratic term can be eliminated on setting
\[
\delta(\beta) = \frac{a(\beta)}{(1 + \beta)^2 + (1 + \beta)}
\]
and thus
\[
y' = -(1 + \beta)y + c(\beta)y^3 + O(y^4)
\]
for some smooth \(c(\beta)\) such that
\[
c(0) = a^2(0) + b(0) = \frac{1}{4} (f_{xx}(0, 0))^2 + \frac{1}{6} f_{xxx}(0, 0)
\]

(d) Finally, by performing the rescaling \(y = \eta/\sqrt{|c(\beta)|}\) obtain the normal form (to cubic order)
\[
\eta' = -(1 + \beta)\eta + \sigma \eta^3
\]
where \(\sigma = \text{sign } c(0) = \pm 1\).