Solutions I
Spring 2019

Solution 1. Damped stochastic oscillator

Fourier transforming the pair of equations
\[
\frac{dX}{dt} = V, \quad m \frac{dV}{dt} = -\gamma V - kX + \sqrt{2D}\xi(t),
\]
with
\[
\tilde{X}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} X(t) dt
\]
e.g., gives
\[
-i\omega \tilde{X}(\omega) = \tilde{V}(\omega), \quad -im\omega \tilde{V}(\omega) = -\gamma \tilde{V}(\omega) + k\tilde{X}(\omega) + \sqrt{2D}\tilde{\xi}(\omega).
\]
Rearranging these equations shows that
\[
(-i\omega(\gamma - im\omega) + k)\tilde{X}(\omega) = \sqrt{2D}\tilde{\xi}(\omega).
\]
It follows that
\[
\tilde{X}(\omega) = G(\omega)\sqrt{2D}\tilde{\xi}(\omega), \quad G(\omega) = \frac{1}{-i\omega(\gamma - im\omega) + k}.
\]
Hence, the power spectrum is
\[
S_X(\omega) = 2D |G(\omega)|^2 = 2D \frac{\frac{1}{i\omega\gamma - m\omega^2 + k}}{\frac{1}{-i\omega\gamma - m\omega^2 + k}} = \frac{2D}{-i\omega\gamma - m\omega^2 + k} \frac{(k - m\omega^2)^2 + \omega^2\gamma^2}{(k - m\omega^2)^2 + \beta^2\omega^2},
\]
\(\omega_0\) is the resonant frequency.

Solution 2. One-dimensional OU process.

Consider the SDE
\[
dX = -kX dt + \sqrt{D}dW(t),
\]
where \(W(t)\) is a Wiener process. Assume the initial condition \(X(0)\) is Gaussian distributed with zero mean and variance \(\sigma^2\). The solution \(X(t)\) is then also Gaussian distributed.

(a) The FP equation for the OU process is
\[
\frac{\partial p(x,t)}{\partial t} = \frac{\partial [kxp(x,t)]}{\partial x} + \frac{D}{2} \frac{\partial^2 p(x,t)}{\partial x^2}.
\]
Taking a fixed (deterministic) initial condition \(X(0) = x_0\), the initial condition of the FP equation is
\[
p(x,0) = \delta(x - x_0).
\]
Introduce the generating or characteristic function using the Fourier transform.

\[ \Gamma(z, t) = \int_{-\infty}^{\infty} e^{izx} p(x, t) dx, \]

We determine a PDE for \( \Gamma \) by Fourier transforming the FP equation. We proceed by using the following results from Fourier analysis: if

\[ \Gamma(z, t) = \int_{-\infty}^{\infty} e^{izx} p(x, t) dx, \quad p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \Gamma(z, t) dz \]

then

\[ -i \frac{\partial \Gamma}{\partial z} = \int_{-\infty}^{\infty} e^{izx} xp(x, t) dx \]

and

\[ \frac{\partial p}{\partial x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-iz) e^{-izx} \Gamma(z, t) dz, \quad \frac{\partial^2 p}{\partial x^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-z^2) e^{-izx} \Gamma(z, t) dz \]

The latter implies that

\[ \int_{-\infty}^{\infty} e^{izx} \frac{\partial^2 p}{\partial x^2} dx = -z^2 \Gamma(z, t), \quad \int_{-\infty}^{\infty} e^{izx} \frac{\partial p}{\partial x} dx = -z \frac{\partial \Gamma}{\partial z}. \]

Hence, Fourier transforming the FP equation

\[ \frac{\partial p}{\partial t} = \frac{\partial [kp]}{\partial x} + D \frac{\partial^2 p}{\partial x^2}, \]

yields

\[ \frac{\partial \Gamma}{\partial t} + kz \frac{\partial \Gamma}{\partial z} = -\frac{D}{2} z^2 \Gamma. \]

Using separation of variables, we set \( \Gamma(z, t) = T(t)Z(z) \) to obtain the ODEs

\[ \frac{T'}{T} = -kz \frac{Z'}{Z} - \frac{D}{2} z^2 = -\lambda \]

for some constant \( \lambda \). These have the solutions

\[ T(t) = e^{-\lambda t}, \quad Z(z) = z^{\lambda/k} e^{-Dz^2/4k}. \]

If we take \( \lambda = nk \) for positive integers \( n \) then the general solution is of the form

\[ \Gamma(z, t) = \sum_{n \geq 0} \gamma_n \left[ z e^{-kt} \right]^n e^{-Dz^2/4k}. \]

The coefficients \( \gamma_n \) are determined by matching the solution with the initial condition \( \Gamma(z, 0) \):

\[ \Gamma(z, 0) = \sum_{n \geq 0} \gamma_n z^n e^{-Dz^2/4k}, \]

that is, the sum over \( n \) can be interpreted as a Taylor expansion with

\[ \sum_{n \geq 0} \gamma_n z^n = \Gamma(z, 0)e^{Dz^2/4k} \equiv \Gamma_0(z). \]
The time-dependent solution can thus be written in the compact form

$$\Gamma(z, t) = \Gamma_0(z e^{-kt}) e^{-Dz^2/4k}.$$  

The initial condition $p(x, 0) = \delta(x - x_0)$ implies that $\Gamma(z, 0) = e^{izx_0}$, which then determines the function $\Gamma_0$:

$$\Gamma_0(z) = e^{izx_0} e^{-Dz^2/4k}.$$  

Hence

$$\Gamma(z, t) = \exp \left[ -\frac{Dz^2}{4k} (1 - e^{-2kt}) + izx_0 e^{-kt} \right].$$

(b) Using the inverse Fourier transform

$$p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\Gamma(z, t)} dz$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz(x - x_0 e^{-kt})} \exp \left[ -\frac{Dz^2}{4k} (1 - e^{-2kt}) \right] dz$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixa} \exp \left[ -\frac{bz^2}{2} \right] dz$$

$$= \frac{1}{\sqrt{2\pi b}} e^{-(x-a)^2/2b}$$

with

$$a = x_0e^{-kt}, \quad b = \frac{D}{2k}[1 - e^{-2kt}].$$

It immediately follows that

$$\langle X(t) \rangle = x_0e^{-kt}, \quad \text{Var}[X(t)] = \frac{D}{2k}[1 - e^{-2kt}].$$

This agrees with part (b) in the case of a deterministic initial condition.

(c) The steady-state FP equation is

$$0 = \frac{\partial[kxp]}{\partial x} + \frac{D}{2} \frac{\partial^2 p}{\partial x^2} = \frac{d}{dx} \left[ kxp + \frac{D}{2} \frac{\partial p}{\partial x} \right].$$

It follows that

$$kxp + \frac{D}{2} \frac{dp}{dx} = \text{constant}$$

Since $p, p' \to 0$ as $x \to \infty$, we see that the constant is zero. Thus

$$kxp + \frac{D}{2} \frac{dp}{dx} = 0,$$

which has the solution

$$p(x) = p_0 e^{-kx^2/D},$$

with $p_0$ determined by the normalization condition. It immediately follows that the steady-state mean and variance are

$$\langle X \rangle = 0, \quad \text{Var}[X] = \frac{D}{2k},$$

which is consistent with (e) in the limit $t \to \infty.$
Solution 3. A planar FP equation.

(a) Consider the planar model

\[ \frac{dx}{dt} = \mu x + y - x(x^2 + y^2), \quad \frac{dy}{dt} = -x + \mu y - y(x^2 + y^2). \]

Multiply the \( \frac{dx}{dt} \) equation by \( x \) and multiply the \( \frac{dy}{dt} \) equation by \( y \):

\[ x \frac{dx}{dt} = \mu x^2 + xy - x^2(x^2 + y^2), \quad y \frac{dy}{dt} = -xy + \mu y^2 - y^2(x^2 + y^2). \]

Adding these two equations and using \( r^2 = x^2 + y^2 \) gives

\[ x \frac{dx}{dt} + y \frac{dy}{dt} = \mu r^2 - r^4. \]

Hence,

\[ \dot{r} = r(\mu - r^2). \]

For \( \mu < 0 \) this has only one real steady-state solution, \( r = 0 \), which is the stable fixed point at the origin. On the other hand, for \( \mu > 0 \) there are two steady-state solutions, an unstable fixed point at the origin (\( r = 0 \)), and \( r = \sqrt{\mu} \). The latter represents a circle of radius \( \sqrt{\mu} \) about the origin in the \( x - y \) plane. This is the limit cycle.

Similarly, multiplying the \( \frac{dx}{dt} \) equation by \( y \) and multiplying the \( \frac{dy}{dt} \) equation by \( x \) gives

\[ y \frac{dx}{dt} = \mu xy + y^2 - xy(x^2 + y^2), \quad x \frac{dy}{dt} = -x^2 + \mu xy - xy(x^2 + y^2). \]

Subtracting the two equations we have

\[ x \frac{dy}{dt} - y \frac{dx}{dt} = -r^2, \]

which implies that

\[ \dot{\theta} = -1. \]

This means that for \( \mu > 0 \) the system rotates at a constant angular frequency \( \omega = -1 \) around the limit cycle of radius \( \sqrt{\mu} \), verifying the occurrence of a Hopf bifurcation.

Solution 4. Rotational diffusion. Consider a Brownian particle undergoing diffusion on the circle \( \theta \in [-\pi, \pi] \). The corresponding FP equation for \( p(\theta,t) \)

\[ \frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial \theta^2}, \quad -\pi < \theta < \pi, \quad p(-\pi,t) = p(\pi,t), \quad p'(-\pi,t) = p'(-\pi,t). \]

where \( D \) is the rotational diffusion coefficient. Suppose that \( p(\theta,0) = \delta(0) \).
(a) Using separation of variables \( p(\theta, t) = \Theta(\theta)T(t) \), we obtain the equations

\[
\frac{1}{D} \frac{dT}{dt} = \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -\mu^2.
\]

The boundary value problem for \( \Theta \) is

\[
\frac{d^2 \Theta}{d\theta^2} = -\lambda \Theta,
\]

which has solutions of the form \( \Theta(\theta) = e^{\pm i\mu \theta} \). Since \( \Theta \) is a periodic function, we require \( \mu = n\pi \) so that we have solutions of the form \( \Theta_n(\theta) = (2\pi)^{-\frac{1}{2}} e^{in\theta} \), \( n \in \mathbb{Z} \). Solving the \( T \) equation

\[
\frac{dT}{dt} = -n^2 \pi^2 DT,
\]

yields \( T(t) = e^{-n^2 \pi^2 Dt} \) so that the general solution is

\[
p(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{in\theta} e^{-n^2 Dt}.
\]

Matching the initial condition \( p(\theta, t) = 0 \) and using the Fourier series representation of the Dirac delta functions implies that \( c_n = 1 \) for all \( n \) and

\[
p(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-n^2 Dt}.
\]

(b) If \( t \) is sufficiently small so that \( Dt \ll \pi^2 \), then \( p(\theta, t) \) is strongly localized around the origin \( \theta = 0 \). This means that the periodic boundary conditions can be ignored and we can effectively take the range of \( \theta \) to be \(-\infty < \theta < \infty\). That is, we introduce the rescalings \( x = \theta/\epsilon \), \( \tau = \epsilon^2 t \) and define \( p(x, \tau) \) according to \( p(\theta, t)d\theta = p(x, \tau)dx \) with

\[
p(x, \tau) = \frac{\epsilon}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\epsilon nx} e^{-\epsilon^2 n^2 D\tau} \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-k^2 D\tau} dk = \frac{1}{\sqrt{4\pi D\tau}} e^{-x^2/4D\tau},
\]

where we have set \( k = \epsilon n \) and taken a continuum limit. It follows that

\[
\langle x^2 \rangle \approx 2D\tau \quad \Rightarrow \quad \langle \theta^2 \rangle = 2Dt,
\]

provided that \( t \ll \pi^2/D \).

(c) In the limit \( t \to \infty \) all phase information is lost and \( p(\theta) = 1/(2\pi) \), that is, one finds a uniform distribution around the circle.

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**Solution 5. Multivariate OU process I.** Consider the Langevin equation

\[
X_i(t + \Delta t) = X_i(t) + \sum_{j=1}^{d} M_{ij} X_j \Delta t + \sum_{j=1}^{d} B_{ij} \Delta W_j(t),
\]
with \( W_i(t) \) an independent Wiener process,

\[
\langle \Delta W_j(t) \rangle, \quad \langle \Delta W_j(t) \Delta W_{j'}(t + n\Delta t) \rangle = \delta_{j,j'}\delta_{n,0}\Delta t.
\]

a) Taking expectations of both sides, we have

\[
\langle X_i(t + \Delta t) \rangle = \langle X_i(t) \rangle + \sum_{j=1}^{d} M_{ij} \langle X_j \rangle \Delta t + \sum_{j=1}^{d} B_{ij} \langle \Delta W_j(t) \rangle.
\]

Since \( \Delta W_i \) has zero mean, we have

\[
\langle X_i(t + \Delta t) \rangle - \langle X_i(t) \rangle = \sum_{j=1}^{d} M_{ij} \langle X_j \rangle \Delta t.
\]

Dividing both sides by \( \Delta t \) and taking the limit \( \Delta t \to 0 \) then gives the first moment equation

\[
\frac{d\overline{X}_i}{dt} = \sum_{j=1}^{d} M_{ij} \overline{X}_j(t),
\]

where \( \overline{X}_i(t) = \langle X_i(t) \rangle \).

b) Introduce the covariance matrix

\[
\Sigma_{ii'}(t) = \langle X_i(t)X_{i'}(t) \rangle - \langle X_i(t) \rangle \langle X_{i'}(t) \rangle.
\]

Now note that

\[
\langle X_i(t + \Delta t)X_{i'}(t + \Delta t) \rangle = \left( \langle X_i(t) \rangle + \sum_{j=1}^{d} M_{ij} \langle X_j \rangle \Delta t + \sum_{j=1}^{d} B_{ij} \langle \Delta W_j(t) \rangle \right) \times \left( \langle X_{i'}(t) \rangle + \sum_{j=1}^{d} M_{i'j'} \langle X_{j'} \rangle \Delta t + \sum_{j'=1}^{d} B_{i'j'} \langle \Delta W_{j'}(t) \rangle \right)
\]

\[
= \langle X_i(t)X_{i'}(t) \rangle + \sum_{j=1}^{d} M_{ij} \langle X_{i'}(t)X_j(t) \rangle \Delta t + \sum_{j'=1}^{d} M_{i'j'} \langle X_i(t)X_{j'}(t) \rangle \Delta t + \sum_{j=1}^{d} \sum_{j'=1}^{d} B_{ij} B_{i'j'} \langle \Delta W_j(t) \Delta W_{j'}(t) \rangle.
\]

and

\[
\langle X_i(t + \Delta t) \rangle \langle X_{i'}(t + \Delta t) \rangle = \left( \langle X_i(t) \rangle + \sum_{j=1}^{d} M_{ij} \langle X_j \rangle \Delta t \right) \left( \langle X_{i'}(t) \rangle + \sum_{j'=1}^{d} M_{i'j'} \langle X_{j'} \rangle \Delta t \right)
\]

\[
= \langle X_i(t) \rangle \langle X_{i'}(t) \rangle + \sum_{j=1}^{d} M_{ij} \langle X_{i'}(t) \rangle \langle X_j(t) \rangle \Delta t + \sum_{j'=1}^{d} M_{i'j'} \langle X_i(t) \rangle \langle X_{j'}(t) \rangle \Delta t.
\]
Subtracting these two equations and using the fact that
\[ \langle \Delta W_j(t) \Delta W_{j'}(t) \rangle = \delta_{j,j'} \Delta t, \]
we obtain the result
\[ \Sigma_{ii'}(t + \Delta t) = \Sigma_{ii'}(t) + \sum_{j=1}^{d} M_{ij} \Sigma_{ij}(t) + \sum_{j'=1}^{d} M_{i'j'} \Sigma_{ij}(t) + \sum_{j=1}^{d} B_{ij} B_{ij'} \Delta t. \]

Rearranging, dividing by \( \Delta t \) and taking the limit \( \Delta t \to 0 \) finally yields
\[ \frac{d \Sigma_{ii'}(t)}{dt} = \sum_{j=1}^{d} M_{ij} \Sigma_{jj'}(t) + \sum_{j'=1}^{d} M_{i'j'} \Sigma_{ij}(t) + \sum_{j=1}^{d} B_{ij} B_{ij'} \]  
\[ (1.1) \]

**Solution 6. Multivariate Ornstein-Uhlenbeck process II.** Consider the multivariate SDE
\[ dX_i = -\sum_{j=1}^{N} A_{ij} X_j dt + \sum_{j=1}^{N} B_{ij} dW_j(t), \]
where \( W_j(t) \) form a set of independent Wiener process:
\[ \langle dW_i(t) \rangle = 0, \quad \langle dW_i(t) dW_j(t') \rangle = \delta_{i,j} \delta(t-t'). \]
Assume a deterministic initial condition \( X_j(0) = \bar{x}_j \).

(a) Write the SDE in matrix form
\[ dX = -AX dt + BdW(t), \]
Performing the change of variables \( Y(t) = e^{At}X(t) \),
\[ dY = e^{At}dX + Ae^{At}X dt = e^{At}BdW(t), \]
Formal integration then shows that
\[ Y(t) = Y(0) + \int_0^t e^{At'}BdW(t'), \]
that is,
\[ X(t) = e^{-At}X(0) + \int_0^t e^{-A(t-t')}BdW(t'). \]

(b) Introduce the correlation function \( C(t,s) = \langle X(t), X^T(s) \rangle \) with components
\[ C_{ij}(t,s) = \langle X_i(t), X_j(s) \rangle = \langle [X_i(t) - \langle X_i(t) \rangle] [X_j(s) - \langle X_j(s) \rangle] \rangle. \]
From part (a),
\[ \langle X(t) \rangle = e^{-At} \langle X(0) \rangle = e^{-At} \bar{x}, \]
since $\langle dW \rangle = 0$ so that
\[ X(t) - \langle X(t) \rangle = \int_0^t e^{-A(t-t')}B dW(t'). \]
It follows that
\[ C(t, s) = \int_0^t e^{-A(t-t')}B dW(t') \int_0^s dW^T(s')B^T e^{-AT(s-s')} \]
\[ = \int_0^t \int_0^s e^{-A(t-t')}B dW(t')dW^T(s')B^T e^{-AT(s-s')} \]
Now
\[ \langle dW(t')dW^T(s') \rangle_{ij} = \langle dW_i(t')dW_j(s') \rangle = \delta_{ij}\delta(t' - s')dt'ds' \]
Therefore,
\[ C(t, s) = \int_0^{\min(t,s)} e^{-A(t-t')}BB^T e^{-AT(t-t')} dt' \]
(c) Introduce the covariance matrix $\Sigma(t) = C(t, t)$ with components
\[ \Sigma_{ij}(t) = \langle [X_i(t) - \langle X_i(t) \rangle][X_j(t) - \langle X_j(t) \rangle] \rangle \]
From part (b), the covariance matrix is given by
\[ \Sigma(t) = \int_0^t e^{-A(t-t')}BB^T e^{-AT(t-t')} dt' \]
Differentiating both sides with respect to $t$ shows that
\[ \frac{d\Sigma}{dt} = \left[ e^{-A(t-t')}BB^T e^{-AT(t-t')} \right]_{t'=t} - \int_0^t A e^{-A(t-t')}BB^T e^{-AT(t-t')} dt' \]
\[ - \int_0^t e^{-A(t-t')}BB^T e^{-AT(t-t')} A^T dt' \]
\[ = -A\Sigma(t) - \Sigma(t)A^T + BB^T \]
If $A$ has distinct eigenvalues $\lambda_j$ then it is diagonalizable, that is, there exists an orthogonal matrix $U$ such that $U^T A U = A_d \equiv \text{diag}(\lambda_1, \ldots, \lambda_N)$. Left-multiplying both sides of the above equation by $U^T$ and right-multiplying by $U$ gives
\[ U^T \frac{d\Sigma}{dt} U = -[U^T A U][U^T \Sigma(t) U] - [U^T \Sigma(t) U][U^T A^T U] + [U^T B U][U^T B^T U], \]
Setting $\hat{\Sigma} = U^T \Sigma U$ and $\hat{B} = U^T B U$, we have
\[ \frac{d\hat{\Sigma}}{dt} = -A_d\hat{\Sigma}(t) - \hat{\Sigma}(t)A_d^T + \hat{B}\hat{B}^T, \]
In component form, we have
\[ \frac{d\hat{\Sigma}_{jk}}{dt} = -(\lambda_j + \lambda_k)\hat{\Sigma}_{jk}(t) + \hat{B}_{jk}. \]
Hence, in the limit $t \to \infty$ we have $\hat{\Sigma}(t) \to \hat{\Sigma}_0$ where

$$(\lambda_j + \lambda_k)\hat{\Sigma}_{0,jk} = \hat{B}_{jk}$$

In original basis, we obtain the stationary covariance matrix $\Sigma_0$ with

$$A\Sigma_0 + \Sigma_0 A^T = BB^T.$$

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Solution 7. Fokker-Planck equation with multiplicative noise.

(a) Consider the steady-state equation

$$0 = \frac{1}{2}\frac{\partial^2}{\partial x^2}[D(x)P],$$

with $\partial_x[D(x)P] = 0$ at $x = \pm 1$. The steady-state FP equation implies that $\partial_x[D(x)P(x)] = 0$, whereas the boundary conditions ensure that the constant is zero. Thus, $D(x)P(x) = A$ for a constant $A$ and

$$P(x) = \frac{A}{D(x)}, \quad A = \int_{-1}^{1} \frac{dx}{D(x)}.$$

If $D(x) = k(b + |x|)$ then

$$A = \frac{1}{k} \int_{-1}^{1} \frac{1}{b + |x|} dx = \frac{1}{k} \int_{0}^{1} \frac{1}{b + x} dx + \frac{1}{k} \int_{-1}^{0} \frac{1}{b - x} dx = \frac{2}{k} \int_{0}^{1} \frac{1}{b + x} dx = \frac{2}{k} \ln \left( \frac{b + 1}{b - 1} \right).$$

For finite $b > 1$, the density is normalizable. In the limit $b \to \infty$, we have $D(x) \to \infty$ and $A \to 0$ such that $P(x)$ becomes the uniform density, consistent with instantaneous diffusion.

(b) The FP equation is now

$$\frac{\partial P}{\partial t} = -a \frac{\partial P}{\partial x} + \frac{b}{2} \frac{\partial^2}{\partial x^2}[xP],$$

with reflecting boundary conditions

$$-aP + \frac{b}{2} \frac{\partial}{\partial x}[xP] = 0, \quad x = 0, 1.$$  

The steady-state equation with reflecting boundary conditions is

$$-aP + \frac{b}{2} \frac{\partial}{\partial x}[xP] = 0$$

for all $x \in [0,1]$. Let $Q(x) = xP(x)$ so that

$$\frac{\partial}{\partial x}[Q] - \frac{2a}{bx} Q = 0.$$
This has a solution of the form
\[ Q(x) = N e^{\frac{2a}{b} \ln x} = N x^{\frac{2a}{b}}, \]
where \( N \) is a normalization factor. Thus
\[ P(x) = N x^{\frac{2a}{b} - 1} \]

(c) Under the change of variables \( y = g(x) \equiv 1/x \), we have
\[ p(y) = P(g^{-1}(y)) \left| \frac{g'(g^{-1}(y))}{g'(y)} \right| = N y^{1 - \frac{2a}{b}} \]
for \( y \in [1, \infty) \). The normalization condition \( \int_{1}^{\infty} p(y) dy \) implies that
\[ N^{-1} = \int_{1}^{\infty} y^{1 - \frac{2a}{b}} dy = -\frac{b}{2a} \left[ y^{-\frac{2a}{b}} \right]_{1}^{\infty} = \frac{b}{2a}. \]
Hence,
\[ p(y) = \frac{2a}{b} y^{1 - \frac{2a}{b}}. \]

We then calculate
\[ \langle 1/x \rangle = \int_{0}^{1} P(x) \frac{dx}{x} = \int_{1}^{\infty} p(y) y dy = \frac{2a}{b} \int_{1}^{\infty} y^{1 - \frac{2a}{b}} dy \]
\[ = \frac{1}{1 - \frac{2a}{b}} \frac{2a}{b} \left[ y^{1 - \frac{2a}{b}} \right]_{1}^{\infty} = \frac{1}{1 - b/2a}, \]
provided that \( b < 2a \).

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Solution 8. FPT in a piecewise linear potential. The MFPT satisfies the ODE
\[ D \frac{d^2 T}{dx^2} + \frac{F(x)}{\gamma} \frac{dT}{dx} = -1, \]
where
\[ F(x) = -U'(x) = \begin{cases} \frac{E_0}{L} & \text{for } -L < x < 0, \\ -\frac{E_0}{L} & \text{for } 0 < x < L \end{cases} \]
The boundary conditions are
\[ T'(-L) = 0, \quad T(L) = 0. \]
First solve equation on domain \( x \in [L, 0) \). We have (after using Einstein relations \( k_B T = \gamma D \)),
\[ \frac{d^2 T}{dx^2} + \frac{E_0}{k_B TL} \frac{dT}{dx} = -\frac{1}{D}, \]
that is,
\[ \frac{d}{dx} \left[ e^{\lambda x} T'(x) \right] = -\frac{1}{D} e^{\lambda x}, \quad \lambda = \frac{E_0}{k_B TL}. \]
Imposing the BC at $x = -L$, gives

$$T'(x) = -\frac{1}{\lambda D} (1 - e^{-\lambda|x+L|}).$$

Integrating again shows that

$$T(x) = C - \frac{1}{\lambda D} \left( x + \lambda^{-1} e^{-\lambda|x+L|} \right), \quad -L < x > 0$$

for some constant $C$. Repeating the analysis on $x \in (0, L)$, we have

$$\frac{d^2T}{dx^2} - \frac{E_0}{k_B T L} \frac{dT}{dx} = -\frac{1}{D},$$

which can be rewritten as

$$\frac{d}{dx}\left[ e^{-\lambda x} T'(x) \right] = -\frac{1}{D} e^{-\lambda x}.$$ 

Integrating once,

$$T'(x) = T'(0^+) e^{\lambda x} - \frac{1}{\lambda D} (e^{\lambda x} - 1).$$

Continuity of $T'(x)$ across $x = 0$ means that

$$T'(0^+) = T'(0^-) = -\frac{1}{\lambda D} (1 - e^{-\lambda L}).$$

Integrating again,

$$T(x) = C' + \frac{1}{\lambda} T'(0^+) e^{\lambda x} - \frac{1}{\lambda D} (\lambda^{-1} e^{\lambda x} - x), \quad 0 < x < L.$$

The constant $C'$ is obtained by imposing the BC $T(L) = 0$:

$$0 = C' - \frac{1}{\lambda^2 D} (1 - e^{-\lambda L}) e^{\lambda L} - \frac{1}{\lambda D} (\lambda^{-1} e^{\lambda L} - L),$$

that is,

$$C' = \frac{2}{\lambda^2 D} e^{\lambda L} - L \frac{\lambda + 1}{\lambda^2 D}.$$

Finally, the constant $C$ is determined by imposing the continuity condition $T(0^-) = T(0^+)$:

$$C = \frac{1}{\lambda^2 D} e^{-\lambda L} + C' + \lambda^{-1} T'(0^+) - \frac{1}{\lambda^2 D} = \frac{2}{\lambda^2 D} (e^{\lambda L} + e^{-\lambda L}) - L \frac{\lambda}{\lambda D} - \frac{3}{\lambda^2 D}.$$

In summary,

$$T(x) = \frac{2}{\lambda^2 D} (e^{\lambda L} + e^{-\lambda L}) - \frac{3}{\lambda^2 D} - \frac{1}{\lambda D} \left( x + L + \lambda^{-1} e^{-\lambda|x+L|} \right)$$

for $-L < x < 0$ and

$$T(x) = \frac{2}{\lambda^2 D} e^{\lambda L} - \frac{1}{\lambda^2 D} - \frac{1}{\lambda D} \left( [L - x] + 2 \lambda^{-1} e^{\lambda x} - \lambda^{-1} e^{-\lambda[L-x]} \right)$$

for $0 < x < L$. 

11