Exercises (lectures 4 and 5)

Problem 8. 1D random walk. Consider the probability distribution for a 1D unbiased random walk

\[ P_N(r) = \frac{1}{2^N} \frac{N!}{\left(\frac{N+r}{2}\right)! \left(\frac{N-r}{2}\right)!}. \]

Using Stirling’s formula

\[ \log N! \approx N \log N - N + \frac{1}{2} \ln(2\pi N) + 1, \]

derive the Gaussian approximation

\[ P_N(r) \sim \frac{1}{\sqrt{2\pi N}} e^{-r^2/2N}. \]

This result includes a factor of 1/2 in order to take into account the fact that \( r \) is even (odd) when \( N \) is even (odd).

Problem 9. Simulation of an OU process. Consider the OU process

\[ \Delta X(t) = -\lambda X(t) \Delta t + \Delta W(t), \quad X(0) = x_0, \]

where \( W(t) \) is a Wiener process. Use direct Euler to simulate 1000 trajectories on the time interval \([0, 1]\) for \( \lambda = 1/2, \Delta t = 0.01 \) and \( x_0 = 1 \). Compare the mean and covariance of the trajectories with the theoretical values.

Problem 10. Multivariate OU process. Consider the Langevin equation

\[ X_i(t + \Delta t) = X_i(t) + \sum_{j=1}^{d} M_{ij} X_j \Delta t + \sum_{j=1}^{d} B_{ij} \Delta W_j(t), \]

with \( W_i(t) \) an independent Wiener process,

\[ \langle \Delta W_j(t) \rangle, \quad \langle \Delta W_j(t) \Delta W_{j'}(t + n\Delta t) \rangle = \delta_{j,j'} \delta_{n,0} \Delta t. \]

a) Taking expectations of both sides, dividing by \( \Delta t \), and taking the limit \( \Delta t \to 0 \), obtain the first moment equation

\[ \frac{d\overline{X}_i(t)}{dt} = \sum_{j=1}^{d} M_{ij} \overline{X}_j(t), \]

where \( \overline{X}_i(t) = \langle X_i(t) \rangle \).

b) Determine \( \langle X_i(t + \Delta t) X_{i'}(t + \Delta) \rangle \) in powers of \( \Delta t \) by carrying out the following steps: (i) use the Langevin equation for \( \langle X_i(t + \Delta t) \rangle \) and \( \langle X_{i'}(t + \Delta) \rangle \); (ii) multiply out all terms; (iii) take averages.
using the mean and variance of the Wiener process together with the conditions \( \langle X_i(t) \Delta W_j(t) \rangle = 0 \) for all \( i,j \). Finally, divide through by \( \Delta t \) and take the limit \( \Delta t \to 0 \) to obtain the equation

\[
\frac{d\Sigma_{ii}(t)}{dt} = \sum_{j=1}^{d} M_{ij} \Sigma_{jj'}(t) + \sum_{j=1}^{d} M_{ij} \Sigma_{jj}(t) + \sum_{j=1}^{d} B_{ij} B_{ij}. 
\]

where

\[
\Sigma_{ij}(t) = \langle X_i(t) X_j(t) \rangle - \langle X_i(t) \rangle \langle X_j(t) \rangle.
\]

**Problem 11. Frequency domain analysis of a simple gene network.** Consider a simple model of protein translation given by the stochastic kinetic equations

\[
\Delta X = (\kappa - \gamma X) \Delta t + q \Delta W_1(t), \quad \Delta Y = (\kappa_p X - \gamma_p Y) \Delta t + q_p \Delta W_2(t),
\]

where \( X(t) \) and \( Y(t) \) are the concentrations of mRNA and protein, \( \gamma, \gamma_p \) are degradation rates, \( \kappa \) is the rate of mRNA production, and \( \kappa_p \) is the rate of protein production. Moreover, \( W_j(t) \) are independent Wiener processes:

\[
\langle \Delta W_i(t) \rangle = 0, \quad \langle \Delta W_i(t) \Delta W_i(t + n \Delta t) \rangle = \delta_{i,j} \delta_{n,0} \Delta t.
\]

(a) By linearizing about the steady-state \( x^* = \kappa / \gamma, \quad y^* = \kappa_p \kappa / (\gamma_p \gamma) \) and using Fourier transforms show that the power spectra of the fluctuations \( \tilde{X}(t) = X(t) - x^* \) and \( \tilde{Y}(t) = Y(t) - y^* \) are given by (after dropping the tildes)

\[
S_{XX}(\omega) = \frac{q}{\omega^2 + \gamma^2}, \quad S_{YY}(\omega) = \frac{q_p}{\omega^2 + \gamma_p^2} + \frac{\kappa_p^2 q}{(\omega^2 + \gamma^2)(\omega^2 + \gamma_p^2)}.
\]

(c) Using the definition of the power spectrum, written in the form

\[
\langle X(t)^2 \rangle = \int_{-\infty}^{\infty} S_{XX}(\omega) \frac{d\omega}{2\pi},
\]

show that

\[
\langle X(t)^2 \rangle = \frac{q}{2\gamma}.
\]

Similarly, show that

\[
\langle Y(t)^2 \rangle = \frac{q_p}{2\gamma_p} + \frac{\kappa_p^2 q}{2\gamma_p \gamma^2} + \mathcal{O}(\gamma^{-3}).
\]

Hint: You should assume that \( \gamma \gg \gamma_p \) and use the result

\[
\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} \frac{d\omega}{2\pi} = \frac{1}{2a}.
\]

(d) It will be shown in lecture 8 that the Fano factors for the number \( M(t) \) of mRNA and number \( N(t) \) of proteins are as follows:

\[
\frac{\text{var}[M]}{\langle M \rangle} = 1, \quad \frac{\text{var}[N]}{\langle N \rangle} = 1 + b,
\]

where \( b = \kappa_p / \gamma \). Use this to determine \( q \) and \( q_p \).