Problems I
Paul C Bressloff (Spring 2019)

Problem 1. Power spectrum of a damped stochastic oscillator. Consider the Langevin equation for a noise-driven, damped harmonic oscillator:

\[ dX(t) = V(t)dt, \quad mdV = [-\gamma V(t)dt + kX(t)]dt + \sqrt{2D}dW(t), \]

where \( W(t) \) is a Wiener process.

(a) By formally setting \( dW(t) = \xi(t)dt \), with \( \langle \xi(t) \rangle = 0 \), \( \langle \xi(t)\xi(t') \rangle = \delta(t-t') \), calculate the spectrum \( S_X(\omega) \) for \( X(t) \).

(b) Plot the spectrum \( S_X(\omega) \) as a function of the angular frequency \( \omega \) for \( \omega_0 \equiv \sqrt{k/m} = 1, 2D/m = 1 \), and various values of \( \beta = \gamma/m \). What happens in the limit \( \beta \to 0 \)? What is the significance of \( \omega_0 \)?

Problem 2. One-dimensional OU process. The FP equation for the OU process is

\[ \frac{\partial p(x,t)}{\partial t} = \frac{\partial [kxp(x,t)]}{\partial x} + \frac{D}{2} \frac{\partial^2 p(x,t)}{\partial x^2}. \]

Taking the fixed (deterministic) initial condition \( X(0) = x_0 \), the initial condition of the FP equation is

\[ p(x,0) = \delta(x-x_0). \]

(a) Introducing the characteristic function (Fourier transform)

\[ \Gamma(z,t) = \int_{-\infty}^{\infty} e^{izx} p(x,t) dx, \]

show that

\[ \frac{\partial \Gamma}{\partial t} + kz \frac{\partial \Gamma}{\partial z} = -\frac{D}{2} z^2 \Gamma. \]

Use separation of variables to obtain a solution of the form

\[ \Gamma(z,t) = \Gamma_0(ze^{-kt})e^{-Dz^2/4k}. \]

with \( \Gamma_0 \) determined by the initial condition for \( p \). Hence, obtain the result

\[ \Gamma(z,t) = \exp \left[ -\frac{Dz^2}{4k}(1 - e^{-2kt}) + izx_0e^{-kt} \right]. \]

[Hint: Set \( \Gamma(z,t) = T(t)Z(z) \) to obtain the ODEs

\[ \frac{T'}{T} = -kz \frac{Z'}{Z} - \frac{D}{2} z^2 = -\lambda \]
for some constant \( \lambda \). Solve these equations for \( \lambda = nk \) and positive integers \( n, n \geq 0 \), and write down the general solution using the principle of linear superposition. Determine the coefficients of the general solution by matching with the initial condition for \( \Gamma(z, 0) \).

(b) The probability density \( p(x, t) \) can be obtained from \( \Gamma(z, t) \) using the inverse Fourier transform

\[
p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \Gamma(z, t) dz.
\]

Substituting for \( \Gamma \) using part (a), show that \( p(x, t) \) is a Gaussian with mean and variance

\[
\langle X(t) \rangle = x_0 e^{-kt}, \quad \text{Var}[X(t)] = \frac{D}{2k}[1 - e^{-2kt}].
\]

(c) Show that the solution to the steady-state FP equation is

\[
p_s(x) = \left(\frac{2\pi D}{k}\right)^{-1/2} e^{-kx^2/2D},
\]

and that this is consistent with the time-dependent solution in the limit \( t \to \infty \).

(d) Consider the Langevin equation for the OU process

\[
dX(t) = -\lambda X(t) dt + dW(t), \quad X(0) = x_0,
\]

where \( W(t) \) is a Wiener process. Use direct Euler to simulate 1000 trajectories on the time interval \([0, 1]\) for \( \lambda = 1/2, \Delta t = 0.01 \) and \( x_0 = 1 \). Compare the mean and covariance of the trajectories with the theoretical values of the OU process.

**Problem 3. A planar FP equation.** Consider the planar dynamical system

\[
\frac{dx}{dt} = A_1(x, y) = \mu x + y - x(x^2 + y^2), \quad \frac{dy}{dt} = A_2(x, y) = -x + \mu y - y(x^2 + y^2).
\]

(a) Transform to polar coordinates \((r, \theta)\) by setting \( x = r \cos \theta, y = r \sin \theta \) and using

\[
\dot{r} = \frac{x \dot{x} + y \dot{y}}{r}, \quad \dot{\theta} = \frac{x \dot{y} - y \dot{x}}{r^2},
\]

Show that the resulting equations become

\[
\frac{dr}{dt} = r(\mu - r^2), \quad \frac{d\theta}{dt} = -1.
\]

Hence show that the system undergoes a Hopf bifurcation with respect to the parameter \( \mu \).

(b) Now consider a stochastic version of the model given by the 2D Langevin equation

\[
dX(t) = A_1(X, Y) dt + \sqrt{2D} dW_1(t), \quad dY(t) = A_2(X, Y) dt + \sqrt{2D} dW_2(t),
\]

where

\[
\langle dW_j(t) \rangle, \quad \langle dW_j(t) dW_{j'}(t') \rangle = \delta_{j,j'} \delta(t - t') dt \, dt'.
\]
The corresponding FP equation for \( p(x, y, t) \) is
\[
\frac{\partial p}{\partial t} = - \frac{\partial}{\partial x} A_1(x, y)p - \frac{\partial}{\partial y} A_2(x, y)p + D \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right).
\]
This can be rewritten in polar coordinates as
\[
\frac{\partial p}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} \left[ \mu (1 - r^2) r^2 p \right] - \frac{\partial}{\partial \theta} p + D \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} \right).
\]
At large times we expect the phase around the limit cycle to be uniformly distributed. Therefore, look for a stationary solution \( p^*(r) \) by setting all time and \( \theta \) derivatives to zero, and solving the resulting ODE for \( p^*(r) \). Show that the solution takes the form
\[
p^*(r) = B \exp(ar^2 - br^4),
\]
and determine the coefficients \( a, b \). How would one determine the constant \( B \)?

(c) Setting \( r = 1 + \rho \) with \( \rho \ll 1 \), show that the behavior of \( p^*(r) \) near the deterministic limit cycle is a Gaussian centered at \( r = 1 \) and has width (standard deviation) \( \sigma = \sqrt{\mu/2D} \).

**Problem 4. Rotational diffusion.** Consider a Brownian particle undergoing diffusion on the circle \( \theta \in [-\pi, \pi] \). The corresponding FP equation for \( p(\theta, t) \)
\[
\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial \theta^2}, \quad -\pi < \theta < \pi, \quad p(-\pi, t) = p(\pi, t), \ p'(-\pi, t) = p'(\pi, t).
\]
where \( D \) is the rotational diffusion coefficient.

(a) Using separation of variables the initial condition \( p(\theta, 0) = \delta(0) \), show that the solution of the FP equation is
\[
p(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-Dn^2t}.
\]
(b) If \( t \) is sufficiently small then \( p(\theta, t) \) is strongly localized around the origin \( \theta = 0 \). This means that the periodic boundary conditions can be ignored and we can effectively take the range of \( \theta \) to be \(-\infty < \theta < \infty \). That is, performing the rescalings \( x = \theta/\epsilon \) and \( \tau = \epsilon^2 t \), show that \( p(\theta, t) \) can be approximated by a Gaussian \( p(x, t) \) and deduce the small-time approximation
\[
\langle \theta^2 \rangle = 2Dt, \quad t \ll \pi^2/D.
\]
(c) What happens in the limit \( t \to \infty \)?

**Problem 5. Multivariate Ornstein-Uhlenbeck process I.** Consider the OU Langevin equation
\[
dX_i(t) = \sum_{j=1}^{d} M_{ij}X_j(t)dt + \sum_{j=1}^{d} B_{ij}dW_j(t),
\]
with $W_i(t)$ an independent Wiener process,
\[ \langle dW_j(t) \rangle = 0, \quad \langle dW_j(t)dW_{j'}(t) \rangle = \delta_{j,j'}dt. \]

(a) Taking expectations of both sides, obtain the first moment equation
\[ \frac{d\overline{X}_i}{dt} = \sum_{j=1}^{d} M_{ij} \overline{X}_j(t), \]
where $\overline{X}_i(t) = \langle X_i(t) \rangle$.

(b) Determine $\langle X_i(t+dt)X_{i'}(t+dt) \rangle$ in powers of $dt$ by carrying out the following steps: (i) use the Langevin equation for $\langle X_i(t+dt) \rangle$ and $X_{i'}(t+dt)$; (ii) multiply out all terms; (iii) take averages using the mean and variance of the Wiener process together with the conditions $\langle X_i(t)dW_j(t) \rangle = 0$ for all $i,j$. Finally, divide through by $dt$ and take the limit $dt \to 0$ to obtain the equation
\[ \frac{d\Sigma_{ii'}(t)}{dt} = \sum_{j=1}^{d} M_{ij} \Sigma_{jj'} + \sum_{j=1}^{d} B_{ij} B_{i'j}. \quad (I.1) \]
where
\[ \Sigma_{ij} = \langle X_i(t)X_j(t) \rangle - \langle X_i(t) \rangle \langle X_j(t) \rangle. \]

**Problem 6. Multivariate Ornstein-Uhlenbeck process II.** Consider the OU Langevin equation
\[ dX_i = \sum_{j=1}^{d} M_{ij} X_j(t) dt + \sum_{j=1}^{d} B_{ij} dW_j(t), \]
with
\[ \langle dW_j(t) \rangle = 0, \quad \langle dW_j(t)dW_{j'}(t') \rangle = \delta_{j,j'}\delta(t-t')dt. \]

(a) Show that the solution in vector form is given by
\[ X(t) = e^{-At} \bar{X} + \int_{0}^{t} e^{-A(t-t')} B dW(t'). \]

(b) Introduce the correlation function $C(t, s) = \langle X(t), X^T(s) \rangle$ with components
\[ C_{ij}(t, s) = \langle X_i(t), X_j(s) \rangle = \langle [X_i(t) - \langle X_i(t) \rangle][X_j(s) - \langle X_j(s) \rangle] \rangle. \]

Using part (a), show that
\[ C(t, s) = \int_{0}^{\min(t,s)} e^{-A(t-t')} BB^T e^{-A^T(s-t')} dt'. \]
(c) Introduce the covariance matrix $\Sigma(t) = \mathbf{C}(t, t)$ with components

$$
\Sigma_{ij}(t) = \langle [X_i(t) - \langle X_i(t) \rangle][X_j(t) - \langle X_j(t) \rangle] \rangle
$$

Derive the matrix equation

$$
\frac{d\Sigma(t)}{dt} = -A \Sigma(t) - \Sigma(t) A^T + \mathbf{B}\mathbf{B}^T.
$$

Hence, show that if $A$ has distinct eigenvalues with positive real part, then $\Sigma(t) \to \Sigma_0$ where $\Sigma_0$ is the stationary covariance matrix satisfying

$$
A \Sigma_0 + \Sigma_0 A^T = \mathbf{B}\mathbf{B}^T.
$$

(d) By formally setting $dW_i(t) = \eta_i(t) dt$ with

$$
\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = \delta_{i,j} \delta(t-t').
$$

show that the spectrum of the $i$th component $X_i(t)$ is

$$
S_i(\omega) = \sum_{j=1}^d \sum_{j'=1}^d \Phi_{ij}^{-1}(\omega) \Phi_{ij'}^{-1}(-\omega) D_{jj'},
$$

where $D_{ij} = \sum_k B_{ik} B_{kj}$ and

$$
\Phi_{ij}(\omega) = -i \omega \delta_{i,j} + M_{ij}.
$$

**Problem 7. 1D Fokker-Planck equation with multiplicative noise.** Consider the Ito Fokker-Planck equation

$$
\frac{\partial P}{\partial t} = -a \frac{\partial P}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} [D(x) P],
$$

with $a$ a constant.

(a) Calculate the steady-state probability density when $a = 0$ and $D(x) = k(b + |x|)$ for constants $k > 0, b > 1$. Take $x \in [-1, 1]$ with reflecting boundary conditions. What happens when $b \to \infty$?

(b) Calculate the steady-state density (up to a normalization factor) when $a \neq 0$ and $D(x) = bx$ for constant $b$. Take $x \in [0, 1]$ with reflecting boundary conditions.

(c) Using part (b), calculate the steady-state density for $y = 1/x$ and determine the normalization factor - use the change of random variables formula from Sect. 1.2. Hence, evaluate $\langle 1/x \rangle$ as a function of $a$ and $b$.

**Problem 8. First passage time problem in a piecewise linear potential.** Find the mean first exit time from the piecewise-linear potential

$$
U(x) = \begin{cases} 
-\frac{E_0}{L^2} x & -L < x < 0 \\
-\frac{E_0}{L^2} x & 0 < x < L
\end{cases}
$$

with a reflecting boundary at $x = -L$ and an absorbing boundary at $x = L$. 
Problem 9. **FPT for a Brownian particle in a semi-infinite domain.** Consider a Brownian particle restricted to a semi-infinite domain $x \in [0, \infty)$ with an absorbing boundary condition at $x = 0$. The FP equation is given by

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}, \quad 0 < x < \infty,$$

with $p(0, t) = 0$.

(a) Check that the solution of the FP equation for the initial condition $x(0) = x_0$ is

$$p(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/4Dt} - \frac{1}{\sqrt{4\pi Dt}} e^{-(x+x_0)^2/4Dt}.$$

(Such a solution can be derived using the method of images, in which one imagines initially placing a fictitious Brownian particle at the image point $x = -x_0$.)

(b) Show that for large times where $\sqrt{Dt} \gg x_0$, the probability density can be approximated by

$$p(x, t) \approx \frac{1}{\sqrt{4\pi Dt}} \frac{x_0}{D t} e^{-(x^2+x_0^2)/4Dt}.$$

(c) Calculate the FPT density $f(x_0, t)$ to reach the origin starting from $x_0$ by calculating the flux through the origin using part (a):

$$f(x_0, t) = D \left. \frac{\partial p(x, t|x_0, 0)}{\partial x} \right|_{x=0}.$$

Hence show that when $\sqrt{Dt} \gg x_0$, we have the asymptotic behavior

$$f(x_0, t) \sim \frac{x_0}{t^{3/2}}.$$

Deduce that the MFPT to reach the origin is infinite.