4.1 Fokker-Planck equation

In the absence of a force, the position $X(t)$ of a Brownian particle evolves according to a Wiener process, and the corresponding distribution of trajectories is determined by the solution to the diffusion equation. Similarly, the position $X(t)$ of a driven Brownian particle evolves according to the Langevin equation (3.2), and the corresponding distribution of trajectories is determined by the solution to a Fokker-Planck equation. Consider a Brownian particle subject to a force $F(x)$, and let $X(t)$ be its position at time $t$. Define the conditional probability density $p(x,t|y,\tau)$ according to

$$p(x,t|y,\tau)\Delta x = \mathbb{P}[x < X(t) < x + \Delta x|X(\tau) = y].$$

The probability density $p(x,t) = p(x,t|y,\tau)$ for the position $x$ of the particle at time $\tau$, given that it started at position $y$ at time $\tau$, satisfies the forward Fokker-Planck (FP) equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial [A(x)p(x,t)]}{\partial x} + D\frac{\partial^2 p(x,t)}{\partial x^2}$$

(4.1)

where $A(x) = F(x)/\gamma$. The 1D FP equation (4.1) can also be rewritten as a probability conservation law according to

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x},$$

(4.2)

where

$$J(x,t) = \frac{1}{\gamma} F(x)p(x,t) - D\frac{\partial p(x,t)}{\partial x}$$

(4.3)

is the probability flux.

One direct way to derive the FP equation is to apply Ito’s lemma to an arbitrary smooth function $f(X(t))$, with $X(t)$ evolving according to equation (3.2).

$$\langle \frac{df(X(t))}{dt} \rangle = \langle A(X(t),t)f'(X(t)) + Df''(X(t)) \rangle$$

$$= \int [A(x,t)f'(x) + Df''(x)] p(x,t)dx,$$

$$= \int f(x) \left[ -\frac{\partial}{\partial x} (A(x,t)p(x,t)) + D\frac{\partial^2}{\partial x^2} (p(x,t)) \right] dx.$$  

(4.4)

after integration by parts, where $p(x,t)$ is the probability density of the stochastic process $X(t)$ under the initial condition $X(t_0) = x_0$. However, we also have

$$\langle \frac{df(X(t))}{dt} \rangle = \left\langle \frac{df(X(t))}{dt} \right\rangle = \frac{d}{dt} \langle f(X(t)) \rangle = \int f(x) \frac{\partial}{\partial t} p(x,t)dx.$$  

(4.5)

Comparing equations (4.4) and (4.5) and using the fact that $f(x)$ is arbitrary, we obtain the FP equation (4.1).
An equilibrium steady-state solution corresponds to the conditions $\frac{\partial p}{\partial t} = 0$ and $J \equiv 0$. This leads to the first-order ODE for the equilibrium density $P(x)$:

$$D \frac{dP(x)}{dx} - \gamma^{-1} F(x) P(x) = 0,$$

which has the solution

$$P(x) = N e^{-U(x)/\gamma D}.$$ 

Here $U(x) = -\int^x F(y)dy$ is a potential energy function and $N$ is a normalization factor (assuming that it exists).

**Einstein relation.** Comparison of the equilibrium distribution with the Boltzmann-Gibbs distribution (7.2) yields the Einstein relation

$$D \gamma = k_B T,$$  \hspace{1cm} (4.6)

where $T$ is the temperature (in degrees Kelvin) and $k_B \approx 1.4 \times 10^{-23} JK^{-1}$ is the Boltzmann constant. This formula relates the variance of environmental fluctuations to the strength of dissipative forces and the temperature. In the case of a sphere of radius $R$ moving in a fluid of viscosity $\eta$, Stoke’s formula can be used, that is, $\gamma = 6\pi \eta R$. For water at room temperature, $\eta \sim 10^{-3} kg m^{-1} s^{-1}$ so that a particle of radius $R = 10^{-9} m$ has a diffusion coefficient $D \sim 100 \mu m^2 s^{-1}$.

**Example 1.** The FP equation for the OU process is

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial [k x p(x,t)]}{\partial x} + D \frac{\partial^2 p(x,t)}{\partial x^2}. \hspace{1cm} (4.7)$$

Given the initial condition $p(x,0) = \delta(x-x_0)$, this has the Gaussian solution

$$p(x,t) = \frac{1}{\sqrt{2\pi D[1-e^{-2kt}]}} e^{-\frac{(x-x_0e^{-kt})^2}{2D[1-e^{-2kt}]}}.$$  \hspace{1cm} (4.8)

Clearly the mean and variance are consistent with the previous calculation. Note that

$$\lim_{t \to \infty} p(x,t) = p_s(x) = \frac{1}{\sqrt{2\pi D / k}} e^{-kx^2/(2D)},$$

which is the stationary probability density.

### 4.2 Higher dimensions

It is straightforward to generalize the above to higher dimensions ($d > 1$). Assuming, for simplicity, isotropic diffusion and friction, equation (3.2) becomes

$$\Delta X_i = \frac{F_i(X)}{\gamma} \Delta t + \sqrt{2D} \Delta W_i(t), \hspace{0.5cm} i = 1, \ldots, d \hspace{1cm} (4.9)$$
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with \( X = (X_1, X_2, \ldots, X_d) \), and

\[
\langle \Delta W_i(t) \rangle = 0, \quad \langle \Delta W_i(t) \Delta W_j(t) \rangle = \delta_{i,j} \Delta t.
\]  

(4.10)

The corresponding Fokker-Planck equation is

\[
\Delta X_i = \frac{F_i(X)}{\gamma} \Delta t + \sqrt{2D} \Delta W_i(t), \quad i = 1, \ldots, d,
\]

(4.11)

with \( X = (X_1, X_2, \ldots, X_d) \), is takes the form

\[
\frac{\partial p(x,t)}{\partial t} = -\frac{1}{\gamma} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} [F_i(x)p(x,t)] + D \nabla^2 p(x,t), \quad \nabla^2 p = \sum_{j=1}^{d} \frac{\partial^2 p}{\partial x_i^2}.
\]

(4.12)

The probability flux is given by the vector \( \mathbf{J} \) with components

\[
J_i(x,t) = F_i(x) \gamma p(x,t) - D \frac{\partial p(x,t)}{\partial x_i}.
\]

(4.13)

The multivariate version of the OU process is given by

\[
\Delta X_i = \sum_{j=1}^{d} M_{ij} X_j \Delta t + \sum_{j=1}^{d} B_{ij} \Delta W_j(t).
\]

(4.14)

and the FP equation is

\[
\frac{\partial p}{\partial t} = -\sum_{i,j=1}^{K} M_{ij} \frac{\partial}{\partial x_j} p(x,t) + \frac{1}{2} \sum_{i,j=1}^{K} D_{ij} \frac{\partial^2 p(x,t)}{\partial x_i \partial x_j},
\]

(4.15)

with \( D_{ij} = \sum_{k} B_{ik} B_{kj} \). Setting \( dW_i(t) = \xi_i(t) dt \) with

\[
\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = \delta_{i,j} \delta(t-t'),
\]

one finds that the spectrum of \( X_i(t) \) is

\[
S_i(\omega) = \sum_{j=1}^{d} \sum_{j'=1}^{d} \Phi_{ij}^{-1}(\omega) \Phi_{ij'}^{-1}(-\omega) D_{jj'},
\]

where \( D_{ij} = \sum_{k} B_{ik} B_{kj} \) and

\[
\Phi_{ij}(\omega) = -i \omega \delta_{i,j} + M_{ij}
\]

is known as the transfer matrix. Finally, one can derive the moment equations

\[
\frac{d\langle X_i \rangle}{dt} = \sum_{j=1}^{d} M_{ij} \langle X_j(t) \rangle,
\]

(4.16)

and

\[
\frac{d\Sigma(t)}{dt} = \mathbf{M} \Sigma(t) + \Sigma(t) \mathbf{M}^T + \mathbf{D},
\]

(4.17)

where \( \mathbf{M}^T \) indicates the matrix transpose of \( \mathbf{M} \), that is, \( M_{ij}^T = M_{ji} \), and

\[
\Sigma_{ij}(t) = \langle X_i(t) X_j(t) \rangle - \langle X_i(t) \rangle \langle X_j(t) \rangle.
\]
4.3 Multiplicative noise

Consider a scalar Langevin equation of the form

\[ dX(t) = A(X)dt + B(X)dW(t). \] (4.18)

(Here we do not necessarily interpret \( X(t) \) as the position of Brownian particle.) A difficulty arises in the interpretation of this equation since, in order to construct a solution of the SDE, we have to deal with stochastic integrals of the form

\[ I(t) = \int_0^t B(X(\tau))dW(\tau). \] (4.19)

Suppose for the moment that \( X(t) \) and \( W(t) \) are deterministic functions of time, and we can apply the theory of Riemann integration. That is, we partition the time interval \([0, t]\) into \( N \) equal intervals of size \( \Delta t \) with \( N\Delta t = t \) and identify the value of the integral with the unique limit (assuming it exists)

\[ I(t) = \lim_{N \to \infty} \sum_{j=0}^{N-1} B([1 - \alpha]X_j + \alpha X_{j+1})\Delta W_j \]

for \( 0 \leq \alpha < 1 \), where \( \Delta W_j = W((j+1)\Delta t) - W(j\Delta t) \) and \( X_j = X(j\Delta t) \). In the deterministic case, the integral is independent of \( \alpha \). Unfortunately, this is no longer true when we have a stochastic integral. One way to see this is to note that the \( \Delta W_j \) are independent random variables. Hence, the function \( b \) is only statistically independent of \( \Delta W_j \) when \( \alpha = 0 \), which is the Itô definition of stochastic integration. On the other hand, when \( \alpha = 1/2 \) we have the Stratonovich version. It turns out that the form of the corresponding FP equation also depends on \( \alpha \), as we now show.

Let us Taylor expand the \( n \)th term in the sum defining the integral \( I(t) \) about the point \( X_n \) and set \( B_n = B(X_n) \):

\[ B((1 - \alpha)X_n + \alpha X_{n+1}) = B_n + \alpha \Delta X_n \frac{\partial B_n}{\partial x} + \frac{1}{2} (\alpha \Delta X_n)^2 \frac{\partial^2 B_n}{\partial x^2} + \ldots, \]

with

\[ \Delta X_n = X_{n+1} - X_n = A_n \Delta t + (B_n + O(\Delta X_n))\Delta W_n. \]

Substituting for \( \Delta X_n \) and dropping terms that are higher order than \( \Delta t \) shows that

\[ B((1 - \alpha)X_n + \alpha X_{n+1}) = B_n + \left( \alpha A_n \frac{\partial B_n}{\partial x} + \frac{\alpha^2 B_n^2}{2} \frac{\partial^2 B_n}{\partial x^2} \right) \Delta t + \left( \alpha B_n \frac{\partial B_n}{\partial x} \right) \Delta W_n. \]

Applying this result to the sum appearing in the definition of the integral, equation (4.19), and again dropping higher order terms in \( \Delta t \) yields the result

\[ \sum_{n=0}^{N-1} B((1 - \alpha)X_n + \alpha X_{n+1}) \Delta W_n = \sum_{n=0}^{N-1} B_n \Delta W_n + \alpha \sum_{n=0}^{N-1} B_n \frac{\partial B_n}{\partial x} (\Delta W_n)^2. \]

Finally, taking the continuum limit with \( dW(t)^2 = dt \), we have

\[ I(t) = \int_0^t B(X(s))dW(s) + \alpha \int_0^t \frac{\partial B(X(s))}{\partial x} B(X(s))ds. \] (4.20)
We can now rewrite the solution in terms of an Ito integral according to

$$X(t) = X_0 + \int_0^t \left[ A(X(s)) + \alpha \frac{\partial B(X(s))}{\partial x} B(X(s)) \right] ds + \int_0^t B(X(s)) dW(s). \quad (4.21)$$

The latter is the solution to an equivalent Ito SDE of the form

$$dX = \left[ A(X) + \alpha B(X) \frac{\partial B(X)}{\partial x} \right] dt + B(X) dW(t). \quad (4.22)$$

Finally, given that we know the FP equation corresponding to an Ito SDE, we can immediately write down the FP equation corresponding to the modified SDE (4.18):

$$\frac{\partial}{\partial t} p(x,t) = - \frac{\partial}{\partial x} \left( [A(x) + \alpha B(x) B'(x)] p(x,t) \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( B(x)^2 p(x,t) \right). \quad (4.23)$$

In summary, the mathematical interpretation of multiplicative noise is ambiguous due to the subtleties of stochastic integration, resulting in different versions of the FP equation.

(a) Ito multiplicative noise arises when carrying out a diffusion approximation of a birth-death or chemical master equation (intrinsic noise). The usual rules of calculus no longer hold, and

$$\langle B(X) dW(t) \rangle = \langle B(X) \rangle \langle dW(t) \rangle = 0.$$ 

Mathematicians use Ito because they can prove theorems! The Ito FP equation is

$$\frac{\partial}{\partial t} p(x,t) = - \frac{\partial}{\partial x} \left( [A(x) + \alpha B(x) B'(x)] p(x,t) \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( B(x)^2 p(x,t) \right). \quad (4.24)$$

(b) Stratonovich multiplicative noise tends to be used when a system is driven by fluctuations in the environment (extrinsic noise). In this case

$$\langle B(X) dW(t) \rangle \neq \langle B(X) \rangle \langle dW(t) \rangle.$$ 

Besides having the advantage that the usual rules of calculus hold, it also has a deeper physical origin. Since noise terms represent at some coarse-grained level the effects of microscopic degrees of freedom that have finite (but short) correlation times, any noise terms should be physically interpreted as the limit in which these correlation times approach zero. This limit produces white noise terms that must be interpreted in the sense of Statonovich. This is shown explicitly below using a backward FP equation. The Stratonovich FP equation is

$$\frac{\partial}{\partial t} p(x,t) = - \frac{\partial}{\partial x} \left( [A(x) p(x,t)] \right) + \frac{1}{2} \frac{\partial}{\partial x} B(x) \frac{\partial}{\partial x} B(x) p(x,t). \quad (4.25)$$

(c) Yet another interpretation of multiplicative noise arises for a system in contact with a thermal bath at constant temperature $T$ and with a space-dependent friction coefficient. The Einstein relation (4.6) then implies that the diffusion coefficient (noise strength) is also space-dependent,
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resulting in a multiplicative noise term in the corresponding SDE. The requirement that the system should approach the equilibrium Boltzmann-Gibbs distribution in the large time limit, leads to the so-called kinetic interpretation of multiplicative noise for which $\alpha = 1$ and the FP equation takes the form.

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial [A(x)p(x,t)]}{\partial x} + \frac{1}{2} \frac{\partial}{\partial x} B(x)^2 \frac{\partial}{\partial x} p(x,t).$$  (4.26)

Consider for example an overdamped Brownian particle with a position-dependent friction coefficient $\Gamma(x)$ such that

$$dX(t) = \Gamma(X)F(X)dt + B(X)dW(t),$$  (4.27)

with $F(X) = -U'(X)$ and the Einstein relation $b(X) = \sqrt{2k_B T \Gamma(X)}$. It is straightforward to show that the steady-state solution of equation (4.26) with $A(x) = -\Gamma(x)U'(x)$ converges to the Boltzmann-Gibbs distribution, that is,

$$\lim_{t \to \infty} p(x,t) \sim e^{-U(x)/k_B T}.$$  

This result would not hold for the other interpretations of multiplicative noise.

**Derivation of the Stratonovich FP equation using a backward equation.**

Let $X(t)$ denote the position of the particle at time $t$, which is taken to evolve according to the SDE

$$dX(t) = [A(X) + \frac{1}{\kappa}B(X)Y(t)]dt,$$  (4.28)

where $Y(t)$ is a stochastic external input evolving according to the OU process

$$dY(t) = -\frac{1}{\kappa} Y(t)dt + \frac{1}{\kappa} dW(t).$$  (4.29)

and $W(t)$ is a Wiener process. Heuristically speaking, in the limit $\kappa \to 0$ we can set $Y(t)dt = \kappa dW(t)$ such that we obtain the scalar SDE (4.18) However, since we have multiplicative noise, there is an ambiguity with regards the interpretation of the noise term from the perspective of stochastic calculus, that is, whether one should choose the Ito or Stratonovich versions. This means that the form of the corresponding FP equation is also ambiguous.

One way to resolve the above issue is to start with the full 2D Fokker-Planck equation and to reduce it to a scalar FP equation in the limit $\epsilon \to \infty$ using an adiabatic reduction. Here we follow an alternative method whose starting point is the backward FP equation. This takes the form

$$\frac{\partial q(x,y,t)}{\partial t} = \left( \frac{1}{\kappa^2} \mathcal{L}_1^* + \frac{1}{\kappa} \mathcal{L}_2^* + \mathcal{L}_3^* \right) q(x,y,t),$$  (4.30)

where

$$\mathcal{L}_1^* = -y \frac{\partial}{\partial y} + \frac{1}{2} \frac{\partial^2}{\partial y^2}, \quad \mathcal{L}_2^* = B(x)y \frac{\partial}{\partial x}, \quad \mathcal{L}_3^* = A(x) \frac{\partial}{\partial x}.$$  (4.31a)

Substituting the following power series expansion,

$$q = q^{(0)} + \kappa q^{(1)} + \kappa^2 q^{(2)} + \ldots$$
into (4.30) yields following hierarchy of equations

\begin{align}
\mathcal{L}_1^* q^{(0)} &= 0 \\
\mathcal{L}_1^* q^{(1)} &= -\mathcal{L}_2^* q^{(0)} \equiv h^{(1)} \\
\mathcal{L}_1^* q^{(2)} &= -\mathcal{L}_2^* q^{(1)} - \mathcal{L}_3^* q^{(0)} + \frac{\partial}{\partial t} q^{(0)} \equiv h^{(2)}.
\end{align}

(4.32a) (4.32b) (4.32c)

It immediately follows that

\[ q^{(0)}(x, y, t) = q(x, t) \]

for some function \( q(x, t) \).

Now observe that the nullspace of \( \mathcal{L}_1 \) is spanned by the stationary density \( p_s(y) \) of \( Y(t) \), which is given by

\[ p_s(y) = \sqrt{\frac{1}{\pi}} e^{-y^2}. \]

(4.33)

Therefore, the righthand side of (4.32b) is orthogonal to the nullspace of \( \mathcal{L}_1 \) since

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^{(1)}(x, y)p_s(y)dx \, dy = \int_{-\infty}^{\infty} y p_s(y) dy = 0. \]

Hence, the Fredholm alternative ensures that (4.32b) is solvable. Indeed, it is straightforward to check that

\[ q^{(1)}(x, y, t) = B(x)y \frac{\partial}{\partial x} q(x, t) \]

(4.34)

solves equation (4.32b). Again appealing to the Fredholm alternative, in order for equation (4.32c) to be solvable, we require

\[ \int_{-\infty}^{\infty} p_s(y) \left\{ \mathcal{L}_2^* q^{(1)} + \mathcal{L}_3^* q - \frac{\partial q}{\partial t} \right\} dy = 0. \]

Now, it is immediate that

\[ \int_{-\infty}^{\infty} p_s(y) \frac{\partial q}{\partial t} \, dy = \int_{-\infty}^{\infty} p_s(y) \mathcal{L}_3^* q \, dy = A(x) \frac{\partial}{\partial x} q, \]

since \( \int_{-\infty}^{\infty} p_s(y) \, dy = 1 \) and \( q \) is independent of \( y \). Furthermore, using (4.34) we have that

\[ \int_{-\infty}^{\infty} p_s(y) \mathcal{L}_2^* q^{(1)} \, dy = \left( \int_{-\infty}^{\infty} y^2 p_s(y) \, dy \right) B(x) \frac{\partial}{\partial x} \left[ B(x) \frac{\partial}{\partial x} q(x, t) \right] \]

\[ = \frac{1}{2} B(x) \frac{\partial}{\partial x} \left[ B(x) \frac{\partial}{\partial x} q(x, t) \right], \]

since \( \int_{-\infty}^{\infty} y^2 p_s(y) \, dy = 1/2 \). Putting this together yields the limiting backward FP equation,

\[ \frac{\partial q}{\partial t} = A(x) \frac{\partial}{\partial x} q + \frac{1}{2} B(x) \frac{\partial}{\partial x} \left[ B(x) \frac{\partial}{\partial x} q(x, t) \right]. \]

The corresponding forward FP equation is given by the Stratonovich version (4.25).