

Name: _____

Score: _____

There are six problems on the front and back of the pages. Each subpart is worth 5 points. Show all of your work where appropriate for full credit.

- 1) Show the following sets of vectors in R^n are linearly dependent or linearly independent. If they are linearly dependent, find a nontrivial linear combination of the vectors which equals zero.

a)

$$u = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}; v = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}; w = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix};$$

$$c_1u + c_2v + c_3w = 0$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Free variable is c_3 . Let $c_3 = 1$, then $c_2 = 2$ and $c_1 = 3$ gives the linear combination

$$3u + 2v + w = 0.$$

b)

$$v_1 = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; v_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}; v_3 = \begin{pmatrix} 3 \\ -4 \\ 2 \\ 1 \\ 0 \end{pmatrix}; v_4 = \begin{pmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix};$$

$$\left[\begin{array}{cccc} 4 & -1 & 3 & -2 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 4 & -1 & 3 & -2 \\ 1 & -1 & 2 & -3 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

This has a 4×4 diagonal submatrix (last four rows) which has nonzero determinant (product of the diagonals is nonzero), so the vectors are linearly independent.

2) Show whether or not the following sets of vectors W form a subspace.

a) W is the set of all vectors in R^4 such that $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$.

Let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) be two vectors in W . Then checking that a linear combination $a(x_1, x_2, x_3, x_4) + b(y_1, y_2, y_3, y_4)$ satisfies the property $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ gives us closure under addition and scalar multiplication as necessary to show W is a subspace.

$$a(x_1, x_2, x_3, x_4) + b(y_1, y_2, y_3, y_4) = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3, ax_4 + by_4)$$

Checking the condition,

$$\begin{aligned}(ax_1 + by_1) + 2(ax_2 + by_2) + 3(ax_3 + by_3) + 4(ax_4 + by_4) &= \\(ax_1 + 2ax_2 + 3ax_3 + 4ax_4) + (by_1 + 2by_2 + 3by_3 + 4by_4) &= \\a(x_1 + 2x_2 + 3x_3 + 4x_4) + b(y_1 + 2y_2 + 3y_3 + 4y_4) &= \\a \cdot 0 + b \cdot 0 &= \\0 &= \end{aligned}$$

W is a subspace.

b) W is the set of all f in \mathcal{F} (the set of all real valued functions defined on the whole real line) such that $f(0) = 0$.

Again we check a linear combination of two vectors from W , f and g .

$$(af + bg)(0) = af(0) + bg(0) = a \cdot 0 + b \cdot 0 = 0$$

W is a subspace, as well.

3) Find a basis for the solution space of the following homogeneous equations.

a)

$$x_1 - 3x_2 - 5x_3 - 6x_4 = 0$$

$$2x_1 + x_2 + 4x_3 - 4x_4 = 0$$

$$x_1 + 3x_2 + 7x_3 + x_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -5 & -6 & 0 \\ 2 & 1 & 4 & -4 & 0 \\ 1 & 3 & 7 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -3 & -5 & -6 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

x_3 is the only free variable which yields through back substitution $x_4 = 0$, $x_3 = t$, $x_2 = -2t$, $x_1 = -t$. Letting $t = 1$ yields the single basis element, $\{(-1, -2, 1, 0)\}$.

b)

$$y^{(4)} - 8y'' + 16y = 0$$

The characteristic equation is $r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r - 2)(r - 2)(r + 2)(r + 2) = 0$. Thus, $r = \pm 2$ are the repeated roots yielding the basis functions $\{e^{2x}, e^{-2x}, xe^{2x}, xe^{-2x}\}$.

4) a) Find two linearly independent solutions to

$$y'' - 2y' + 2y = 0. \quad (1)$$

The characteristic equation $r^2 - 2r + 2 = 0$ yields the roots $r = 1 \pm i$, which correspond to the two solutions $y_1 = e^x \cos(x)$ and $y_2 = e^x \sin(x)$.

b) Verify that they are linearly independent using the Wronskian.

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^x \cos(x) & e^x \sin(x) \\ -e^x \sin(x) + e^x \cos(x) & e^x \cos(x) + e^x \sin(x) \end{vmatrix} \\ &= e^x \cos(x)(e^x \cos(x) + e^x \sin(x)) - e^x \sin(x)(-e^x \sin(x) + e^x \cos(x)) \\ &= e^{2x}(\cos^2(x) + \sin^2(x)) \\ &= e^{2x} \neq 0 \end{aligned}$$

c) Find a particular solution to (1) with the initial conditions $y(0) = 2$ and $y'(0) = 3$. The general solution is $y = C_1 e^x \cos(x) + C_2 e^x \sin(x)$ so that $y(0) = C_1 = 2$. Since $y'(x) = C_1(-e^x \sin(x) + e^x \cos(x)) + C_2(e^x \cos(x) + e^x \sin(x))$, we get that $y'(0) = C_1 + C_2 = 3$, so that $C_2 = 1$ and the particular solution is

$$y(x) = 2e^x \cos(x) + e^x \sin(x).$$

5) Find the general solution to the forced oscillatory pendulum equation without damping

$$Lx'' + gx = F_0 \cos(\omega t)$$

a) when the forcing is not resonant.

1) Solving the homogeneous equation $Lx'' + gx = 0$.

We get the characteristic equation $Lr^2 + g = 0$, so that $r = \pm\sqrt{\frac{g}{L}}i$. From this we get the general complementary solution

$$y_c(t) = C_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + C_2 \sin\left(\sqrt{\frac{g}{L}}t\right)$$

2) Solving for the particular solution.

Since we are in the nonresonance case we know that the natural frequency $\sqrt{\frac{g}{L}} = \omega_0$ is not equal to ω , so that we take our trial particular solution to look like

$$y_p(t) = A \cos(\omega t) + B \sin(\omega t).$$

We now plug into the equation

$$x'' + \omega_0^2 x = F_0/L \cos(\omega t) \tag{2}$$

to solve for A and B using the method of undetermined coefficients. We get

$$(-A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)) + \omega_0^2(A \cos(\omega t) + B \sin(\omega t)) = F_0/L \cos(\omega t)$$

Equating coefficients of the $\sin(\omega t)$ and $\cos(\omega t)$ we get the two equations

$$A(\omega_0^2 - \omega^2) = F_0/L$$

$$B(\omega_0^2 - \omega^2) = 0$$

which yields

$$A = \frac{F_0/L}{\omega_0^2 - \omega^2}$$

$$B = 0$$

3) The general solution is then

$$y(t) = y_c + y_p = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0/L}{\omega_0^2 - \omega^2} \cos(\omega t)$$

b) when the forcing is resonant.

The homogeneous solution remains the same. Now since $\omega_0 = \omega$, we need the trial solution multiplied by the independent variable

$$y_p(t) = t(A \cos(\omega t) + B \sin(\omega t))$$

Plugging this into the differential equation (2) (remembering that $\omega_0 = \omega$) we get

$$2(-A\omega \sin(\omega t) + B\omega \cos(\omega t)) = F_0/L \cos(\omega t)$$

Again, equating coefficients and solving yields $A = 0$, $B = \frac{F_0/L}{2\omega}$, which gives the final solution

$$y(t) = y_c + y_p = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0/L}{2\omega} t \sin(\omega t)$$

6) a) Find the eigenvalues and eigenvectors of the matrix.

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

To find the eigenvalues we solve for λ the characteristic equation $|A - \lambda I| = 0$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 2 & 1 - \lambda \end{vmatrix} = -\lambda(1 - \lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

Therefore the eigenvalues are $\lambda = -1, 2$. Finding the associated eigenvectors requires solving the equation $(A - \lambda I)v = 0$ for the vector v .

For $\lambda = -1$,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

so that v_2 is a free variable. Letting $v_2 = 1$ forces $v_1 = -1$ and gives the eigenvector $(-1, 1)$. For $\lambda = 2$,

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

so that v_2 is a free variable. Letting $v_2 = 2$ forces $v_1 = 1$ and gives the eigenvector $(1, 2)$.

b) Solve the following system of first order equations

$$\begin{aligned} x' &= y \\ y' &= 2x + y \end{aligned}$$

for the general solution. Differentiating the first equation yields the second order equation in x

$$x'' = y' = 2x + y = x' + 2x$$

or

$$x'' - x' - 2x = 0$$

Solving the characteristic equation gives the roots $r = -1, 2$ and the general solution $x(t) = C_1 e^{-t} + C_2 e^{2t}$. Again referring to the first equation we get the solution for $y(t) = -C_1 e^{-t} + 2C_2 e^{2t}$.