Test #1

Math 2250 Summer 2003

Name:________________ Codename:________________ Score: ________

Each problem is worth 10 points. Show all of your work where appropriate for full credit.

1) Solve the following first order differential equations

a) $\frac{dy}{dx} = e^x$; $y(0) = 1$ Integrating up yields $y(x) = e^x + C$. The initial condition implies $C = 0$.

b) $\frac{dy}{dx} = y^2x - yx$; $y(0) = 1$ Separating variables yields

$$\frac{1}{y(y-1)}\ dy = x\ dx$$

$$\int -\frac{1}{y} + \frac{1}{y-1}\ dy = \int x\ dx$$

$$-\ln(y) + \ln(y-1)\ dy = x^2 + C$$

$$-\ln(y) + \ln(y-1)\ dy = x^2 + C$$

$$1 - \frac{1}{y} = Ce^{x^2}$$

$C = 0$

$$y(x) = 1$$

c) $\frac{dy}{dx} + \frac{1}{x}y = \sin(x)$; $y(\pi) = 0$ Using an integrating factor $e^{\int \frac{1}{x}\ dx} = x$ we get the differential equation

$$(xy)' = x\sin(x)$$

Integrating up yields

$$xy = \int x\sin(x)\ dx + C = -x\cos(x) + sin(x) + C$$

having integrated by parts. $C = -\pi$ so that the particular solution is

$$y(x) = -\cos(x) + \frac{1}{x}\sin(x) - \frac{\pi}{x}$$

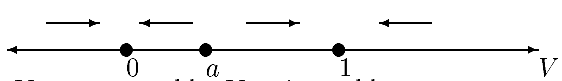
2) The following autonomous ODE describes an excitable medium (i.e. cardiac tissue or a forest) without recovery.

$$\frac{dV}{dt} = V(1-V)(V-a)\ 0 < a < \frac{1}{2}$$

a) In general, what is a critical point?
A critical point of an autonomous ODE $\frac{dy}{dx} = f(y)$ are the y-values such that $f(y) = 0$. At these y-values then $\frac{dy}{dx} = 0$ so that the solution doesn't change (i.e. the solution is a fixed point).

b) What are the critical points of the given autonomous ODE?
$V(1-V)(V-a) = 0$ for $V = 0, a, 1$.

c) Plot the phase line and determine the stability of the critical points.



$V = 0$ - stable; $V = a$ - unstable; $V = 1$ - stable

d) Why is $V = a$ considered a threshold?
$V = a$ is a threshold, because if we start just below $V = a$, then we go to $V = 0$. However, if we start just above $V = a$, then we go to $V = 1$.

e) (Extra Credit) Find the implicit general solution of the given ODE. Separating variables and integrating yields

$$\int \frac{1}{V(1-V)(V-a)}\, dV = \int dt + C$$

$$\int \frac{-1/a}{V} + \frac{1/(1-a)}{1-V} + \frac{1/(a(1-a))}{V-a}\, dV = t + C$$

$$-1/a \ln(V) - 1/(1-a)\ln(1-V) + 1/(a(1-a))\ln(V-a) = t + C$$

$$\frac{V^{1/(a(1-a))}}{V^{1/a}(1-V)^{1/(1-a)}} = Ce^t$$

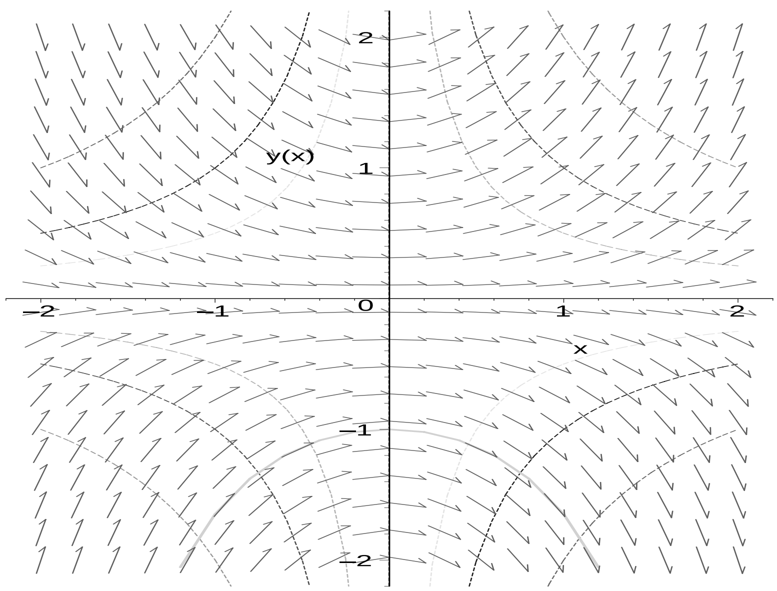
3)

$$\frac{dy}{dx} = xy \;; y(0) = -1$$

a) Prove the existence and uniqueness of the given IVP.

Since $f(x,y) = xy$ is continuous for all values of x and y it is continuous in a neighborhood of the initial value, and so we are guaranteed existence by the existence/uniqueness theorem. Since $\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} xy = x$, is also continuous for all values of x and y including the initial value, we are also guaranteed uniqueness by the existence/uniqueness theorem.

b) Plot the direction field for the given ODE, and sketch the unique solution of the IVP.



Solid curve is the solution of the IVP. The dashed curves are isoclines.

c) Use Euler's Method to approximate y at $x = 1$ using a stepsize of $h = 0.5$.

x	$y \ (y_{n+1} = y_n + h(x_n y_n))$
$x_0 = 0$	$y_0 = -1$
$x_1 = 0.5$	$y_1 = -1 + 0.5(0(-1)) = -1$
$x_2 = 1$	$y_2 = -1 + 0.5(0.5(-1)) = -1.25$

4) a) If it exists, find the inverse of the given matrix.

i) $A = \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix}$

$$|A| = 1 \cdot 8 - (-2) \cdot (-4) = 0$$

so A is not invertible.

ii) $B = \begin{bmatrix} 2 & 7 & 4 \\ 1 & 3 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

We do elementary row operations to check the determinant, but since we would need to do these operations to find the inverse anyway, we augment the matrix at the start.

$$[B|I] = \left[\begin{array}{ccc|ccc} 2 & 7 & 4 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 7 & 4 & 1 & 0 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \overset{-2R_1+R_2 \to R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \overset{-2R_1+R_3 \to R_3}{\longrightarrow}$$

Here we notice that we have used elementary row ops. to reduce B to a triangular matrix with nonzero diagonal elements so that the determinant of B is nonzero and the inverse (which we're in the process of finding) exists. Note that the fact the determinant is nonzero is not affected by the row swap. Why not?

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \overset{-2R_3+R_1 \to R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 5 & -2 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \overset{-3R_2+R_1 \to R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 11 & -2 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

So

$$B^{-1} = \begin{bmatrix} -3 & 11 & -2 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

iii) $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ -2 & 3 & -4 \end{bmatrix}$

Since $-R_2 = R_3$, performing the row op. $R_2 + R_3 \to R_3$ yields a zero row, and thus, a zero determinant implying no inverse exists.

b) Find the solution set of the given linear system.

i) $$\begin{aligned} x_1 - 2x_2 &= 0 \\ -4x_1 + 8x_2 &= 0 \end{aligned}$$

From 4a)i) we know that the coefficient matrix is not invertible so we either have no solutions or infinitely many solutions, but this is also a homogeneous system, so that we must have an infinite number of solutions. Reducing the system and letting $x_2 = t$ a free variable, we get the solution set is $(x_1 = 2t, x_2 = t)$ for $t \in R$.

ii) $$\begin{aligned} 2x + 7y + 4z &= 0 \\ x + 3y + 2z &= 1 \\ 2x + 6y + 5z &= 4 \end{aligned}$$

From 4a)ii) we know that the inverse of the coefficient matrix exists so our solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= B^{-1} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{bmatrix} -3 & 11 & -2 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \end{aligned}$$

iii) $$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + 3x_2 - 4x_3 &= 2 \\ 2x_1 - 3x_2 + 4x_3 &= 3 \end{aligned}$$

Again, since no inverse exists (from 4a)iii)), we know we either have no solutions or infinitely many solutions. Performing the row op. $R_2 + R_3 \to R_3$ yields the inconsistent system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -2 & 3 & -4 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Thus, there is no solution.