4.1

2) $a = (-1, 0, 2)\ ; b = (3, 4, -5)$

$$|a - b| = |(-1, 0, 2) - (3, 4, -5)| = |(-4, -4, 7)| = \sqrt{(-4)^2 + (-4)^2 + 7^2} = 9$$

$$2a + b = 2(-1, 0, 2) + (3, 4, -5) = (1, 4, -1)$$

$$3a - 4b = 3(-1, 0, 2) - 4(3, 4, -5) = (-15, -16, 26)$$

12) $w = (2, -2)\ ; u = (4, 1)\ ; v = (-2, -1)$. Find c_1, c_2 such that $w = c_1 u + c_2 v$. Solve the linear system

$$\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

We find the inverse of the coefficient matrix as

$$-\frac{1}{2} \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$$

so

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{1}{2} \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Then $w = 3u + 5v$.

24) Find a, b, c such that $au + bv + cw = 0$ for $u = 1, 4, 5)\ ; v = (4, 2, 5)\ ; w = (-3, 3, 1)$. If $a = b = c = 0$, then u, v, w are linearly independent, otherwise they are dependent.

$$\begin{bmatrix} 1 & 4 & -3 \\ 4 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmenting the coefficient matrix and reducing to reduced echelon form produces the identity matrix augmented by the zero vector. Thus, $a = b = c = 0$, and u, v, w are linearly independent. Notice that we can perform the elementary row operations without augmenting by the homogeneous right hand side until the end. Why?

32) $V = \{(x, y, z) \in R^3 | z = 2x + 3y\}$ Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in V$. Closure under '+':

$$\begin{aligned} z_1 + z_2 &= 2x_1 + 3y_1 + 2x_2 + 3x_2 \\ &= 2x_1 + 2x_2 + 3y_1 + 3y_2 \\ &= 2(x_1 + x_2) + 3(y_1 + y_2) \end{aligned}$$

So V is closed under '+'. Closure under scalar multiplication:

$$\begin{aligned} cz_1 &= c(2x_1 + 3y_1) \\ &= 2(cx_1) + 3(cy_1) \end{aligned}$$

So V is closed under scalar multiplication, and V is a subspace.

36) $V = \{(x, y, z) \in R^3 | xyz = 1\}$ Check scalar multiplication. $c(x, y, z)\ (cx)(cy)(cz) = c^3(xyz) = c^3 \cdot 1 \neq 1$, so closure under scalar multiplication fails, and V is not a subspace.

40) As shown in class, if u, v, w are linearly dependent vectors then there exist scalars a, b, c not all zero such that $au + bv + cw = 0$. Assume $c = 0$. Then $au + bv = 0$ with either a or b not zero. However, this contradicts the fact that u, v are linearly independent and the only way for $c_1 u + c_2 v = 0$ is if $c_1 = c_2 = 0$. Therefore, $c \neq 0$. Since, $c \neq 0$ we can divide by c to get $w = -a/cu - b/cv$, which is a linear combination of u and v.

4.2

4) $W = \{(x_1, x_2, x_3) \in R^3 | x_1 + x_2 + x_3 = 1\}$ Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in W$. Checking closure under addition.

$$\begin{aligned} (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\ &= 1 + 1 \\ &= 2 \neq 1 \end{aligned}$$

So closure under addition fails, and W is not a subspace.

8) $W = \{(x_1, x_2) \in R^2 | x_1^2 + x_2^2 = 0\}$ Let $(x_1, x_2), (y_1, y_2) \in W$. Checking closure under scalar multiplication.

$$\begin{aligned} (cx_1)^2 + (cx_2)^2 &= c^2(x_1^2) + c^2(x_2^2) \\ &= c \cdot 0 + c \cdot 0 \\ &= 0 + 0 = 0 \end{aligned}$$

So W is closed under scalar multiplication. Checking closure under addition.

$$\begin{aligned}(x_1+y_1)^2+(x_2+y_2)^2 &= (x_1^2+2x_1y_1+y_1^2)+(x_2^2+2x_2y_2+y_2^2)\\ &= (x_1^2+y_1^2)+(x_2^2+y_2^2)+2x_1y_1+2x_2y_2\\ &= 2x_1y_1+2x_2y_2\end{aligned}$$

$2x_1y_1+2x_2y_2=0$ only if $x_1$or y_1 and x_2 or y_2 is zero. The question then is is this ever not true? No. The way to satisfy $x_1^2+x_2^2=0$ is for $x_1=x_2=0$. Therefore, closure under addition does indeed hold, and W is a subspace. In fact we have shown that $W=\{0\}$ (which we know is a subspace).

18) We put the linear homogeneous system into reduced row echelon form as

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3\\ 0 & 1 & 0 & 1 & 4\\ 0 & 0 & 1 & 2 & -5\end{bmatrix}$$

Thus, $x_4=s$ and $x_5=t$ are free variables, so that

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=s\begin{pmatrix}2\\-1\\-2\\1\\0\end{pmatrix}+t\begin{pmatrix}3\\-4\\5\\0\\1\end{pmatrix}$$

20) We put the linear homogeneous system into reduced row echelon form as

$$\begin{bmatrix} 1 & 0 & 0 & 5\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & 2\end{bmatrix}$$

Thus, $x_4=t$ is a free variable, so that

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=t\begin{pmatrix}-5\\3\\-2\\1\end{pmatrix}$$

28) Show $S=\{x\in R^n|Ax=kx, A\in R^{n\times n}, k \text{ constant}\}$ is a subspace. Let $x,y\in S$. Then

$$\begin{aligned}A(x+y) &= Ax+Ay\\ &= kx+ky\\ &= k(x+y)\end{aligned}$$

So that S is closed under addition. Also for c a scalar,

$$\begin{aligned}A(cx) &= c(Ax)\\ &= c(kx)\\ &= k(cx)\end{aligned}$$

So that S is also closed under scalar multiplication, and S is a subspace.