2.3 10) Falling with linear air resistance is modeled as a force balance by the ODE

$$\frac{dv}{dt} = g - \rho v$$

where v is the velocity, g is acceleration due to gravity, and ρ is the drag coefficient. Before opening the parachute the IVP is

$$v' = -32 - 0.15v$$

with $v(0) = 0$ and $y(0) = 10,000$ft where y is the altitude above the ground. Solving this ODE yields the particular solution (using either integrating factors or separation of variables) $v(t) = 213.333(e^{-0.15t} - 1)$ and $v(20) = -202.7$ft/sec. Integrating up the velocity equation yields the position, $y(t) = 11422.2 - 1422.22e^{-0.15t} - 213.333t$ and $y(20) = 7084.75$. The velocity and position after 20 sec without the parachute become the initial conditions when the parachute is deployed to give us the new IVP

$$v' = -32 - 1.5v$$

with $v(0) = -202.7$ and $y(0) = 7084.75$. Solving this ODE for velocity yields $v(t) = -21.333 - 181.379e^{-1.5t}$. Integrating this up to get position yields $y(t) = 6964.83 + 120.919e^{-1.5t} - 21.333t$. We can use this formula to solve for when $y = 0$. Numerically solving this equation for t yields $t = 326.5 sec \approx 5.44 min$. ($v(326.5) \approx 21.33$ft/sec $\approx$ 15 mph).

12) The weight of the barrel is -640lbs. The mass is 20 slugs. The buoyancy force, $B = 62.5 lbs/ft^3 \cdot 8 ft^3 = 500 lbs$. The force due to water resistance, $F_R = 1 lb/(ft/sec) \cdot v$. This leaves us with the ODE

$$20v' = -640 + 500 - v$$

or

$$v' = -7 - 1/20v$$

. Solving for v yields $v(t) = 140(e^{-1/20t} - 1)$. The time is $20\ln(28/13) \approx 15.35$ when $v = -75$. Integrating up the velocity equation we get position $y(t) = 2800(1 - e^{-1/20t}) - 140t$ so that $y(15.35) \approx 648.31$ft. Any deeper than this and the barrel will pick up too much velocity and explode on impact with the ocean floor.

26) Since we're given an initial condition in terms of $v = 0$ and $r = r_0$, it makes sense to formulate the given ODE

$$\frac{dv}{dt} = -\frac{GM}{r^2}$$

as an ODE for v in terms of the independent variable r (keeping in mind our final goal of a relationship between r and t). We do this by appealing to the chain rule such that $\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$, so that the ODE we'll work with is

$$v\frac{dv}{dr} = -\frac{GM}{r^2}.$$

We can separate variables to solve the ODE to get

$$v\ dv = -\frac{GM}{r^2}\ dr$$

which we integrate up yielding the implicit relationship between v and r,

$$\frac{1}{2}v^2 = \frac{GM}{r} + C.$$

Using $v(r_0) = 0$, we get the particular solution,

$$\frac{1}{2}v^2 = \frac{GM}{r} - \frac{GM}{r_0}$$

Realizing that we're only interested in negative velocities (because we're falling) we can solve explicitly for v to get $v = \frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{r_0}}$, which gives us an ODE for r in terms of t. Since the right hand side is autonomous, we try separation of variables.

$$\sqrt{\frac{rr_0}{r_0 - r}}\ dr = -\sqrt{2GM}\ dt$$

$$\sqrt{r_0}\int\sqrt{\frac{r}{r_0 - r}}\ dr = -\int\sqrt{2GM}\ dt + C = -\sqrt{2GM}t + C$$

To solve the left integral we appeal to the hint and use the change of variables $r = r_0\cos^2(\theta); dr = -2r_0\cos(\theta)\sin(\theta)\, d\theta$. Making this change of variables the left integral becomes

$$\begin{aligned}
\sqrt{r_0}\int\sqrt{\frac{r}{r_0 - r}}\, dr &= -\sqrt{r_0}\int\sqrt{\frac{\cos^2(\theta)}{1-\cos^2(\theta)}}2r_0\cos(\theta)\sin(\theta)\, d\theta \\
&= -2(r_0)^{3/2}\int\cos^2(\theta)\, d\theta \\
&= -(r_0)^{3/2}(\cos(\theta)\sin(\theta) + \theta)
\end{aligned}$$

we get the last step from the table of integrals in the back of the book. Plugging this back into the equation and changing variables back to r gives us

$$-(r_0)^{3/2}(\cos(\cos^{-1}(\sqrt{\frac{r}{r_0}})\sin(\cos^{-1}(\sqrt{\frac{r}{r_0}}) + \cos^{-1}(\sqrt{\frac{r}{r_0}}) = -\sqrt{2GM}t + C$$

$$-(r_0)^{3/2}(\sqrt{\frac{r}{r_0}}\sqrt{\frac{r_0 - r}{r_0}}) + \cos^{-1}(\sqrt{\frac{r}{r_0}}) = -\sqrt{2GM}t + C$$

$$\sqrt{\frac{r_0}{2GM}}(\sqrt{rr_0 - r^2} + r_0\cos^{-1}(\sqrt{\frac{r}{r_0}})) = t + C.$$

Using $r(0) = r_0$ forces $C = 0$. This is the equation we wanted to show. b) Taking $G \approx 6.7 \times 10^{-11}(\mathrm{N(m/kg)}^2)$, $M \approx 6 \times 10^{24}$kg, $r_0 = R + 1 \times 10^6$m where R is the radius of the earth in meters ($R = 6.4 \times 10^6$m), we wish to find the time when $r = R$. Plugging into our equation yields a time of $t \approx 510$sec. Plugging the positions into the velocity equation yields

$$v = -\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{r_0}} = -4120\mathrm{m/s}$$

2.4

4) The solution of the IVP $y' = x - y; y(0) = 1$ is $y(x) = x - 1 + 2e^{-x}$.

x	$y_{(0.1)}$	$y_{(0.05)}$	y
0.0500		0.9500	0.9525
0.1000	0.9000	0.9050	0.9097
0.1500		0.8648	0.8714
0.2000	0.8200	0.8290	0.8375
0.2500		0.7976	0.8076
0.3000	0.7580	0.7702	0.7816
0.3500		0.7467	0.7594
0.4000	0.7122	0.7268	0.7406
0.4500		0.7105	0.7253
0.5000	0.6810	0.6975	0.7131

14) The solution of the IVP $xy' = y^2; y(1) = 1$ is $y(x) = -\frac{1}{\ln(x)-1}$.

x	$y_{(0.01)}$	$y_{(0.005)}$	y	$(y - y_{(0.005)})/y$
1.2000	1.2215	1.2222	1.2230	$0.0008/1.2230 = 0.0006$
1.4000	1.5026	1.5048	1.5071	$0.0023/1.5071 = 0.0015$
1.6000	1.8761	1.8814	1.8868	$0.0054/1.8868 = 0.0029$
1.8000	2.4020	2.4138	2.4259	$0.0121/2.4259 = 0.0050$
2.0000	3.2031	3.2304	3.2589	$0.0285/3.2589 = 0.0087$

23) The y-values at $x = 1$ for the different h-values are

x	$y_{(0.1)}$	$y_{(0.1)}$	$y_{(0.1)}$	$y_{(0.1)}$
1	1.2262	1.2300	1.2306	1.2307

2.6

4)

0.25	0.80762	0.80760
0.50	0.71309	0.71306

14)

x	$y_{(0.2)}$	$y_{(0.1)}$	y	$y - y_{(0.1)}/y$
1.0	1.0	1.0	1.0	0
1.2	1.222957	1.222973	1.222975	0.000001
1.4	1.507040	1.507092	1.507096	0.000003
1.6	1.886667	1.886795	1.886805	0.000005
1.8	2.425586	2.425903	2.425928	0.000010
2.0	3.257946	3.258821	3.258891	0.000021

23) For $h = 0.24$, $y(1) \approx 1.230735$, while for $h = 0.1, 0.05, 0.025$, $y(1) \approx 1.230731$ to 6 decimal places.