

1.5 6) To solve the ODE $xy' + 5y = 7x^2$: Dividing through by x we get

$$y' + \frac{5}{x}y = 7x$$

and notice that this ODE has y appearing linearly and so we use an integrating factor to solve it. We attempt to write the left hand side as a total derivative by multiplying each side by the integrating factor, $e^{\int 5/x dx} = e^{5 \ln(x)} = x^5$. This gives us

$$x^5 y' + 5x^4 y = 7x^6$$

for the ODE. Left side is now the total derivative, $\frac{d}{dx}(x^5 y)$. Integrating up the ODE yields

$$\int \frac{d}{dx}(x^5 y) = \int 7x^6 + C$$

$$x^5 y = x^7 + C.$$

Our general solution is then $y = x^2 + Cx^{-5}$. Using the initial condition, $y(2) = 5$, we get that $C = 32$, so that our particular solution is $y = x^2 + 32x^{-5}$.

24) To solve the ODE $(x^2 + 4)y' + 3xy = x$: Dividing through by $(x^2 + 4)$ we get

$$y' + \frac{3x}{x^2 + 4}y = \frac{x}{x^2 + 4}$$

and notice that this ODE has y appearing linearly and so we use an integrating factor to solve it. We attempt to write the left hand side as a total derivative by multiplying each side by the integrating factor,

$$\begin{aligned} e^{\int \frac{3x}{x^2+4} dx} &= e^{\frac{3}{2} \ln(x^2+4)} \\ &= (x^2 + 4)^{3/2} \end{aligned}$$

. This gives us

$$\frac{d}{dx}((x^2 + 4)^{3/2}y) = x\sqrt{x^2 + 4}$$

for the ODE. Integrating up the ODE yields

$$(x^2 + 4)^{3/2}y = \frac{1}{3}(x^2 + 4)^{3/2} + C$$

so that our general solution is

$$y = \frac{1}{3} + \frac{C}{(x^2 + 4)^{3/2}}$$

Using the initial condition, $y(0) = 1$, we get that $C = \frac{16}{3}$, so that our particular solution is

$$y = \frac{1}{3} + \frac{16}{3(x^2 + 4)^{3/2}}$$

34) Our general ODE for coupled reservoirs is

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

where x is the amount of the pollutant (in millions of ft^3), r_i and r_o are the inflow and outflow rates, respectively, and c_i and c_o are the inflow and outflow concentrations, respectively. There is a constant volume of $V = 8$ billion ft^3 so that $c_o = x/8000$. The rates are fixed at $r_i = r_o = 500$ million ft^3/day . The concentration flowing in, $c_i = 0.0005$ and the initial concentration in the lake is 0.0025 which implies that $x(0) = 8000 \cdot 0.0025 = 20\text{ft}^3$. So our ODE looks like

$$\frac{dx}{dt} = 5000.0005 - 500x/8000 = 1/4 - 1/16x$$

We can use an integrating factor to solve this ODE for the general solution

$$x(t) = 4 + Ce^{-t/16}$$

Using the initial condition we find the particular solution

$$x(t) = 4 + 16e^{-t/16}.$$

We use this to find out when $x = 0.001 * 8000 = 8$.

$$8 = 4 + 16e^{-t/16}$$

$$t = -16 \ln(1/4) \approx 22.2 \text{ days.}$$

- 38) a) The flow rate into tank 1 is 0 and out of tank 1 is 5 gal/min. With the volume in tank 1 fixed at 100 gal, we can write down the rate of change of the amount of salt in the tank, $x(t)$, as

$$\frac{dx}{dt} = -5x/100$$

so that the amount of salt is $x(t) = Ce^{-1/20t}$ or $x(t) = 50e^{-1/20t}$ considering the initial conditions.

- b) Since the outflow rate from tank 1 is the inflow rate to tank 2, the amount of salt in tank 2, $y(t)$ follows the ODE

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}.$$

Since we know how x depends explicitly on time, we can plug that in to get an ODE for y alone.

$$\frac{dy}{dt} = \frac{5}{2}e^{-1/20t} - \frac{5y}{200}.$$

Solving this using another integrating factor yields

$$y(t) = 150e^{-t/40} - 100e^{-t/20}$$

having used the initial condition $y(0) = 50$.

- c) The maximum will occur when

$$y' = -\frac{15}{4}e^{-1/40t} + 5e^{-1/20t} = \frac{5}{4}e^{-1/40t}(3 - 4e^{-1/40t}) = 0.$$

The maximum occurs at $t = 40 \ln(3/4) \approx 11.5$ days.

- 2.1 4) To solve the ODE $x' = 7x(x - 13)$ we use separation of variable and partial fraction decomposition on the resulting integral. Separating variables yields

$$\frac{1}{x(x - 13)} dx = 7 dt.$$

Integrating yields

$$\int \frac{1}{x(x - 13)} dx = \int 7 dt + C = 7t + C.$$

Using partial fraction decomposition on the left hand side integrand we get

$$\begin{aligned} \frac{1}{x(x - 13)} &= \frac{A}{x} + \frac{B}{x - 13} \\ &= \frac{A(x - 13)}{x(x - 13)} + \frac{Bx}{x(x - 13)} \\ &= \frac{A(x - 13)}{x(x - 13)} + \frac{Bx}{x(x - 13)} \\ &= \frac{(A + B)x - 13A}{x(x - 13)} \end{aligned}$$

so that $(A+B)$ must be 0 and $-13A$ must be 1. The left hand side integral looks like $-\frac{1}{13} \int \frac{1}{x} - \frac{1}{x-13} dx = -\frac{1}{13}(\ln(x) - \ln(x - 13))$ Solving for x implicitly yields

$$\frac{x}{x - 13} = Ce^{-91t}$$

Using the initial condition here gives us $C = 17/4$. Solving explicitly for x yields $x = \frac{221}{17 - 4e^{91t}}$.

- 8) The time rate of change of an alligator population (in dozens) being proportional to the square of the population means the alligator population evolves according to the differential equation

$$\frac{dP}{dt} = kP^2$$

where k is the constant of proportionality which we will figure out given the two data points (0,1) and (10,2) (letting $t = 0$ at 1988). Solve the ODE using separation of variables to get the population function $P(t) = \frac{1}{-kt+C}$ using the initial condition $P(0) = 1$ means $C = 1$. Since $P(10) = \frac{1}{-k \cdot 10 + 1} = 2$, $k = \frac{1}{20}$, so that the population evolves according to $P(t) = \frac{20}{20-t}$. We get to 4 dozen alligators when $t = 15$ and the population explodes as $t \rightarrow 20$.

18) The logistic equation is solved for as

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

where P_0 is the initial population, M is the carrying capacity, and k is a rate of growth. Given that the population is in thousands so that $M = 100$ and $P_0 = 50$ and we know that $P'(0) = 1$ we can find k .

$$P'(t) = \frac{MP_0(M - P_0)(kM)e^{-kMt}}{(P_0 + (M - P_0)e^{-kMt})^2}$$

So

$$P'(0) = kP_0(M - P_0) = 1$$

and $k = 0.0004$. Looking for the time at $P = 80$ yields the equation

$$80 = \frac{100}{1 + e^{-0.04t}}$$

so that $t = 25 \ln(1/4) \approx 34.7$ days.

2.2 10) The critical point of the autonomous differential equation

$$\frac{dx}{dt} = 7x - x^2 - 10$$

occur for the x -values where $7x - x^2 - 10 = (-x + 5)(x - 2) = 0$ which are $x = 2, 5$. For $x < 2$, $\frac{dx}{dt} < 0$. For $2 < x < 5$, $\frac{dx}{dt} > 0$. For $x > 5$, $\frac{dx}{dt} < 0$. This implies that $x = 2$ is unstable and $x = 5$ is stable. Solving the ODE

$$\int \frac{1}{(x-5)(x-2)} dx = - \int dt$$

$$\int \frac{1}{x-5} - \frac{1}{x-2} dx = -3t + C$$

$$\ln\left(\frac{x-5}{x-2}\right) = -3t + C$$

$$\frac{x-5}{x-2} = Ce^{-3t}$$

. For a general initial condition $x(0) = x_0$ we get $C = \frac{x_0-5}{x_0-2}$. Solving explicitly for x yields

$$x(t) = \frac{2(5-x_0) + 5(x_0-2)e^{3t}}{(5-x_0) + (x_0-2)e^{3t}}$$

14) The ODE of a logistic population with harvesting is

$$x' = kx(M - x) - hx.$$

The critical points are found by setting the right hand side to 0.

$$kx(M - x) - hx = kx\left(M - \frac{h}{k} - x\right) = 0$$

so that the critical points are $x = 0$ and $x = M - \frac{h}{k}$. a) If $h < kM$, we see that for $x < 0$, $x' < 0$; for $0 < x < M - \frac{h}{k}$, $x' > 0$; for $x > M - \frac{h}{k}$, $x' < 0$, so that $x = 0$ is unstable and $x = M - \frac{h}{k}$ is the new stable limiting population of the logistic dynamics. b) However, if $kM < h$, then the critical point $x = M - \frac{h}{k} < 0$ and is unstable, while $x = 0$ is now stable. In this parameter range if we start with any positive population, it will become extinct from over harvesting.