

- 1.1 8) If $y_1 = \cos(x) - \cos(2x)$ and $y_2 = \sin(x) - \cos(2x)$, then $y_1' = -\sin(x) + 2\sin(2x)$, $y_1'' = -\cos(x) + 4\cos(2x)$ and $y_2' = \cos(x) + 2\sin(2x)$, $y_2'' = -\sin(x) + 4\cos(2x)$. Plugging into the left hand side of the ODE $y_1'' + y_1 = 3\cos(2x)$ we get

$$y_1'' + y_1 = -\cos(x) + 4\cos(2x) + (\cos(x) - \cos(2x)) = 3\cos(2x)$$

and

$$y_2'' + y_2 = -\sin(x) + 4\cos(2x) + (\sin(x) - \cos(2x)) = 3\cos(2x).$$

So y_1 and y_2 solve the ODE $y'' + y = 3\cos(2x)$.

- 16) Substituting $y = e^{rx}$ into the ODE $3y'' + 3y' - 4y = 0$ yields the equation

$$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0.$$

Dividing this equation by (the nonzero) e^{rx} yields a quadratic equation for r

$$3r^2 + 3r - 4 = 0$$

which we solve by the quadratic equation to get $r = \frac{3 \pm \sqrt{57}}{6}$.

- 18) We check that $y(x) = Ce^{2x}$ satisfies the ODE $y' = 2y$. Taking the derivative of y we get $y' = 2Ce^{2x}$, so we see that y' is exactly $2y$. To satisfy the initial condition $y(0) = 3$, $y(0) = Ce^{2 \cdot 0} = C \cdot 1 = C = 3$, and C must be 3.
- 24) We check that $y(x) = x^3(C + \ln(x))$ satisfies the ODE $xy' - 3y = x^3$. Taking the derivative of y we get $y' = 3x^2(C + \ln(x)) + x^3(\frac{1}{x}) = 3x^2(C + \ln(x)) + x^2$, so we see that

$$\begin{aligned} xy' - 3y &= x(3x^2(C + \ln(x)) + x^2) - 3(x^3(C + \ln(x))) \\ &= 3x^3C + 3x^3\ln(x) + x^3 - 3x^3C - 3x^3\ln(x) \\ &= x^3, \end{aligned}$$

which is exactly the right hand side. To satisfy the initial condition $y(1) = 17$, $y(0) = 1^3(C + 0) = 17$, and C must be 17.

- 28) The slope of the line through the two points (x, y) and $(x/2, 0)$ is given by

$$y' = \frac{y - 0}{x - x/2} = \frac{2y}{x}$$

so that the function which has a tangent line at (x, y) and goes through $(x/2, 0)$ also solves the ODE $2y' = x$.

- 32) The time rate of change of a population P being proportional to the square root of the population translates to the equation

$$\frac{dP}{dt} = k\sqrt{P}$$

where k is the constant of proportionality and has units of 1/time.

- 1.2 6) To find the general solution to $y' = x\sqrt{x^2 + 9}$ we integrate up both sides so that $y = \int(x\sqrt{x^2 + 9} dx) + C$ If we make the substitution $u = x^2 + 9$ so that $du = 2dx$, then $\int(x\sqrt{x^2 + 9} dx = \frac{1}{2} \int \sqrt{u} du = \frac{1}{5}u^{3/2}$. From this we get that $y = \frac{1}{3}(x^2 + 9)^{3/2} + C$. We use $y(-4) = 0$ to get that $C = -\frac{125}{3}$ and the particular solution is $y = \frac{1}{3}(x^2 + 9)^{3/2} - \frac{125}{3}$.
- 8) To find the general solution to $y' = \cos(2x)$ we integrate up both sides so that $y = \int \cos(2x) dx + C = \frac{1}{2} \sin(2x) + C$. We use $y(0) = 1$ to get that $C = 1$ and the particular solution is $y = \frac{1}{2} \sin(2x) + 1$.
- 14) To find the position function $x(t)$ from acceleration we integrate up twice, first to get velocity and then to get position.

$$a(t) = v'(t)$$

so that

$$v(t) = \int a(t) dt + C = \int 2t + 1 dt + C = t^2 + t + C.$$

Since $v(0) = -7$, $C = -7$. Now

$$v(t) = x'(t)$$

so that

$$x(t) = \int v(t) dt + C = \int t^2 + t - 7 dt + C = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + C.$$

Since $x(0) = 4$, $C = 4$, so that the particular solution is

$$x(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + 4.$$

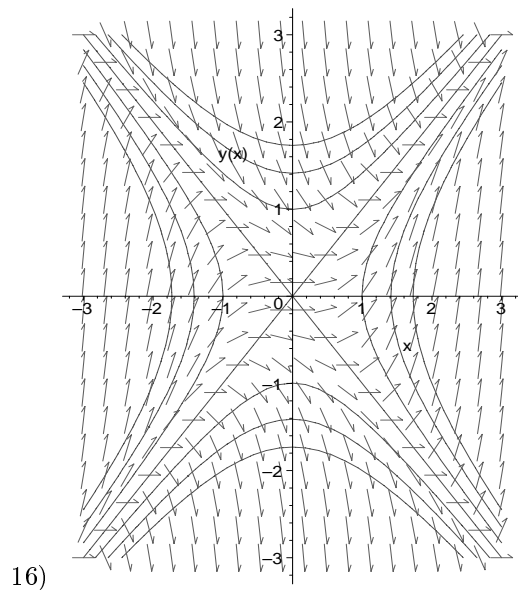
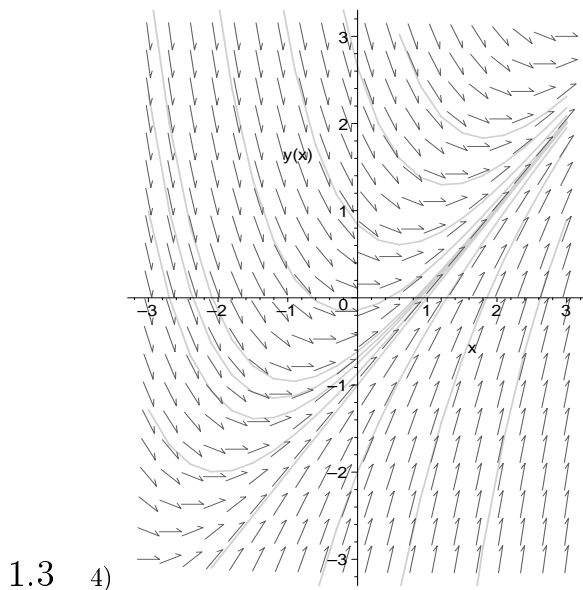
- 20) To find how long it takes a ball dropped (with 0 initial velocity) from a 400ft high building to hit the ground we consider finding position from a constant gravity acceleration. First we integrate to find the velocity function $v(t) = -32t$. Integrating the velocity equation to get position yields $x(t) = -16t^2 + 400$. The ball hits the ground when $x = -16t^2 + 400 = 0$ or when $t = 5$ sec. At this time the velocity $v(5) = -32(5) = -160$ ft/sec.
- 36) Considering the preamble to Example 4 and the velocity of the river is $v_R = v_0 \left(1 - \frac{x^4}{a^4}\right)$ we get the ODE

$$y' = \frac{v_0}{v_S} \left(1 - \frac{x^4}{a^4}\right).$$

We can integrate this up to get

$$y = \frac{v_0}{v_S} \left(x - \frac{1}{5} \frac{x^5}{a^4}\right) + C$$

Starting from the left bank gives us the condition $y(-a) = 0$, so that $C = \frac{v_0}{v_S} \frac{4}{5} a$. When the swimmer reaches the other side ($x = a$), he is $y(a) = \frac{v_0}{v_S} \frac{8}{5} a$ down stream. Using $v_0 = 9$ mi/h, $v_S = 3$ mi/h, and $a = \frac{1}{2}$, we get that $y(\frac{1}{2}) = 2.4$ mi.



22) Since $f(x, y) = x \ln(y)$ is continuous on the rectangle given by $\{-\infty < x < \infty, 0 < y\}$ and the initial condition $y(1) = 1$ falls into this rectangle, the ODE $y' = f(x, y)$ is guaranteed a solution exists. Furthermore since $\frac{\partial f}{\partial y} = \frac{x}{y}$ is continuous everywhere except when $y = 0$, we're also guaranteed a unique solution for a while.

28) Since $f(x, y) = \frac{x-1}{y}$ is continuous on the two rectangles given by $\{-\infty < x < \infty, 0 < y\}$ and $\{-\infty < x < \infty, y < 0\}$, but the initial condition $y(1) = 0$ falls between these rectangles, the ODE $y' = f(x, y)$ is not guaranteed a solution exists.

32) $y(x) = 0$ is a solution for all x -values. We can solve this ODE by separation of variables to get the solution $y(x) = x^3$ (as discussed in class). Since the $\frac{\partial}{\partial y} 3y^{2/3} = 2y^{-1/3}$ is not continuous at $y = 0$, the theorem does not guarantee us uniqueness. Thus, the two solutions do not conflict with the theorem.

CPA a) Plugging in $y = ax + b$ into the ODE $y' = \sin(x - y)$ yields the equation

$$a = \sin((1 - a)x + b).$$

Since the left hand side is constant, the right hand side must also be constant. This is only true when $(1 - a) = 0$ or $a = 1$. This leaves us with the equation

$$1 = \sin(b)$$

to be solved for b . This yields $b = n\pi/2$ for $n = \dots, -2, -1, 0, 1, 2, \dots$

b) No, we would need $C \rightarrow \infty$ for the initial condition $y(\pi/2) = 0$ to satisfy $y(x) = x - 2 \tan^{-1} \left(\frac{x-2-C}{x-C} \right)$.

CPB Similar arguments to CPA.

1.4 2) Separating the variables of the ODE $y' + 2xy^2 = 0$ yields the equation

$$\frac{1}{y^2} dy = -2x dx$$

which we integrate up to get

$$-\frac{1}{y} = -x^2 + C.$$

Solving for y explicitly we get $y = \frac{1}{x^2 + C}$ (C has gone to $-C$ because it is a general constant).

12) Separating the variables of the ODE $yy' = x(y^2 + 1)$ yields the equation

$$\frac{y}{y^2 + 1} dy = x dx$$

which we integrate up (using u -substitution) to get

$$\frac{1}{2} \ln(y^2 + 1) = \frac{1}{2} x^2 + C.$$

Solving for y^2 we get the implicit general solution $y^2 = Ce^{x^2} - 1$.

20) Separating the variables of the ODE $y' = 3x^2(y^2 + 1)$ yields the equation

$$\frac{1}{y^2 + 1} dy = 3x^2 dx$$

which we integrate up (using $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}$) to get

$$\tan^{-1}(y) = x^3 + C.$$

Solving for y we get the explicit general solution $y = \tan(x^3 + C)$. Using the initial condition $y(0) = 1$, we get that $C = \tan^{-1}(1) = \pi/4$. The particular solution is then $y(x) = \tan(x^3 + \pi/4)$.

26) Separating the variables of the ODE $y' = 2xy^2 + 3x^2y^2$ yields the equation

$$\frac{1}{y^2} dy = 2x + 3x^2 dx$$

which we integrate up to get

$$-\frac{1}{y} = x^2 + x^3 + C.$$

Solving for y we get the explicit general solution $y = -\frac{1}{x^2 + x^3 + C}$. Using the initial condition $y(1) = -1$, we get that $1 = \frac{1}{2+C}$, so that $C = -1$. The particular solution is then $y(x) = -\frac{1}{x^2 + x^3 - 1}$.

42) Given the initial value problem for barometric pressure in terms of altitude, x , $p' = -0.2p$ with initial condition $p(0) = 29.92$, we are asked to find the pressure at 10,000ft and at 30,000ft. We separate variables to solve for p and get that $p(x) = 29.92e^{-0.2x}$. Plugging in our two altitudes gives us $p(10,000ft = 1.9mi) = 29.92e^{-0.2 \cdot 1.9} \approx 20.5$ and $p(30,000ft = 5.7mi) = 29.92e^{-0.2 \cdot 5.7} \approx 9.6$. If people cannot survive at $p < 15$ then solving for x at this p -value tells us the altitude at which this occurs (should be between 10,000ft and 30,000ft from our previous calculation). Solving for x in $15 = 29.92e^{-0.2x}$ yields $x \approx 3.45mi \approx 18,200ft$.

CP Consider the ODE

$$\frac{dx}{dt} = \frac{m}{n}x - \frac{1}{n}x^2.$$

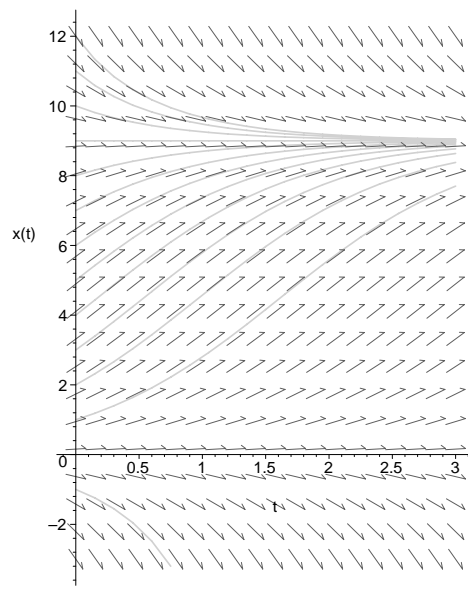
Letting $m = 9$ and $n = 7$, we get the ODE

$$\frac{dx}{dt} = \frac{1}{7}(9x - x^2).$$

As $t \rightarrow \infty$ it appears that the population approaches 9.

b) Maple gives the general solution as $x(t) = 9\frac{1}{1+9Ce^{-9/7t}}$. As $t \rightarrow \infty$, $x \rightarrow 9$ as expected.

c) We ask the question how long does it take for a population to grow by 80%? Letting $x(0) = x_0$ we get the particular population $x(t) = \frac{9x_0}{x_0 + (9-x_0)e^{-9/7t}}$. The population is $x = 0.8x_0$ when $t = -7/9 \ln(\frac{0.8x_0}{9-x_0})$. We can now plug in whatever initial population we like to get a specific t .



a)