

- 1.1 8) If  $y_1 = \cos(x) - \cos(2x)$  and  $y_2 = \sin(x) - \cos(2x)$ , then  $y_1' = -\sin(x) + 2\sin(2x)$ ,  $y_1'' = -\cos(x) + 4\cos(2x)$  and  $y_2' = \cos(x) + 2\sin(2x)$ ,  $y_2'' = -\sin(x) + 4\cos(2x)$ . Plugging into the left hand side of the ODE  $y_1'' + y_1 = 3\cos(2x)$  we get

$$y_1'' + y_1 = -\cos(x) + 4\cos(2x) + (\cos(x) - \cos(2x)) = 3\cos(2x)$$

and

$$y_2'' + y_2 = -\sin(x) + 4\cos(2x) + (\sin(x) - \cos(2x)) = 3\cos(2x).$$

So  $y_1$  and  $y_2$  solve the ODE  $y_1'' + y_1 = 3\cos(2x)$ .

- 16) Substituting  $y = e^{rx}$  into the ODE  $3y'' + 3y' - 4y = 0$  yields the equation

$$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0.$$

Dividing this equation by (the nonzero)  $e^{rx}$  yields a quadratic equation for  $r$

$$3r^2 + 3r - 4 = 0$$

which we solve by the quadratic equation to get  $r = \frac{3 \pm \sqrt{57}}{6}$ .

- 18) We check that  $y(x) = Ce^{2x}$  satisfies the ODE  $y' = 2y$ . Taking the derivative of  $y$  we get  $y' = 2Ce^{2x}$ , so we see that  $y'$  is exactly  $2y$ . To satisfy the initial condition  $y(0) = 3$ ,  $y(0) = Ce^{2 \cdot 0} = C \cdot 1 = C = 3$ , and  $C$  must be 3.
- 24) We check that  $y(x) = x^3(C + \ln(x))$  satisfies the ODE  $xy' - 3y = x^3$ . Taking the derivative of  $y$  we get  $y' = 3x^2(C + \ln(x)) + x^3(\frac{1}{x}) = 3x^2(C + \ln(x)) + x^2$ , so we see that

$$\begin{aligned} xy' - 3y &= x(3x^2(C + \ln(x)) + x^2) - 3(x^3(C + \ln(x))) \\ &= 3x^3C + 3x^3\ln(x) + x^3 - 3x^3C - 3x^3\ln(x) \\ &= x^3, \end{aligned}$$

which is exactly the right hand side. To satisfy the initial condition  $y(1) = 17$ ,  $y(0) = 1^3(C + 0) = 17$ , and  $C$  must be 17.

- 28) The slope of the line through the two points  $(x, y)$  and  $(x/2, 0)$  is given by

$$y' = \frac{y - 0}{x - x/2} = \frac{2y}{x}$$

so that the function which has a tangent line at  $(x, y)$  and goes through  $(x/2, 0)$  also solves the ODE  $2y' = x$ .

- 32) The time rate of change of a population  $P$  being proportional to the square root of the population translates to the equation

$$\frac{dP}{dt} = k\sqrt{P}$$

where  $k$  is the constant of proportionality and has units of 1/time.

- 1.2 6) To find the general solution to  $y' = x\sqrt{x^2 + 9}$  we integrate up both sides so that  $y = \int(x\sqrt{x^2 + 9} dx) + C$  If we make the substitution  $u = x^2 + 9$  so that  $du = 2dx$ , then  $\int(x\sqrt{x^2 + 9} dx = \frac{1}{2} \int \sqrt{u} du = \frac{1}{5}u^{3/2}$ . From this we get that  $y = \frac{1}{3}(x^2 + 9)^{3/2} + C$ . We use  $y(-4) = 0$  to get that  $C = -\frac{125}{3}$  and the particular solution is  $y = \frac{1}{3}(x^2 + 9)^{3/2} - \frac{125}{3}$ .
- 8) To find the general solution to  $y' = \cos(2x)$  we integrate up both sides so that  $y = \int \cos(2x) dx + C = \frac{1}{2} \sin(2x) + C$ . We use  $y(0) = 1$  to get that  $C = 1$  and the particular solution is  $y = \frac{1}{2} \sin(2x) + 1$ .
- 14) To find the position function  $x(t)$  from acceleration we integrate up twice, first to get velocity and then to get position.

$$a(t) = v'(t)$$

so that

$$v(t) = \int a(t) dt + C = \int 2t + 1 dt + C = t^2 + t + C.$$

Since  $v(0) = -7$ ,  $C = -7$ . Now

$$v(t) = x'(t)$$

so that

$$x(t) = \int v(t) dt + C = \int t^2 + t - 7 dt + C = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + C.$$

Since  $x(0) = 4$ ,  $C = 4$ , so that the particular solution is

$$x(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + 4.$$

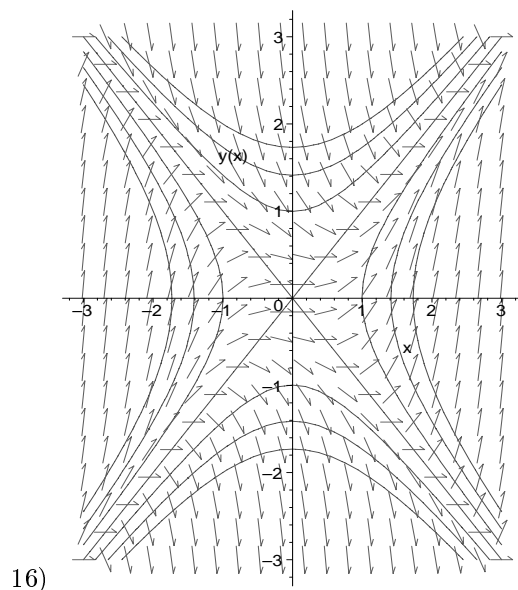
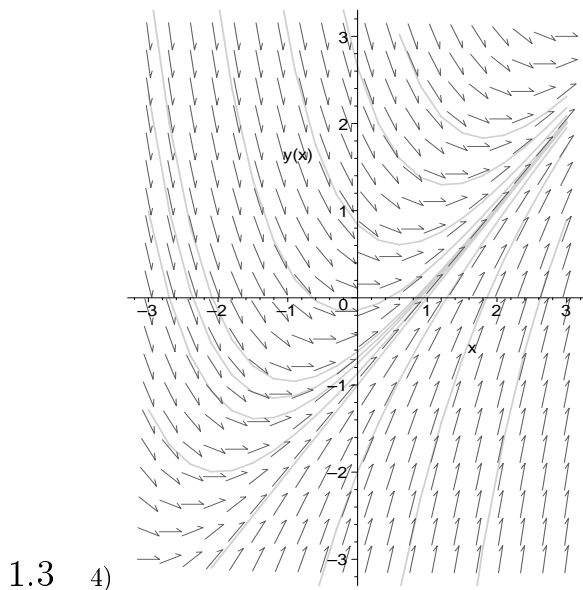
- 20) To find how long it takes a ball dropped (with 0 initial velocity) from a 400ft high building to hit the ground we consider finding position from a constant gravity acceleration. First we integrate to find the velocity function  $v(t) = -32t$ . Integrating the velocity equation to get position yields  $x(t) = -16t^2 + 400$ . The ball hits the ground when  $x = -16t^2 + 400 = 0$  or when  $t = 5$ sec. At this time the velocity  $v(5) = -32(5) = -160$ ft/sec.
- 36) Considering the preamble to Example 4 and the velocity of the river is  $v_R = v_0 \left(1 - \frac{x^4}{a^4}\right)$  we get the ODE

$$y' = \frac{v_0}{v_S} \left(1 - \frac{x^4}{a^4}\right).$$

We can integrate this up to get

$$y = \frac{v_0}{v_S} \left(x - \frac{1}{5} \frac{x^5}{a^4}\right) + C$$

Starting from the left bank gives us the condition  $y(-a) = 0$ , so that  $C = \frac{v_0}{v_S} \frac{4}{5}a$ . When the swimmer reaches the other side ( $x = a$ ), he is  $y(a) = \frac{v_0}{v_S} \frac{8}{5}a$  down stream. Using  $v_0 = 9$ mi/h,  $v_S = 3$ mi/h, and  $a = \frac{1}{2}$ , we get that  $y(\frac{1}{2}) = 2.4$ mi.



22) Since  $f(x, y) = x \ln(y)$  is continuous on the rectangle given by  $\{-\infty < x < \infty, 0 < y\}$  and the initial condition  $y(1) = 1$  falls into this rectangle, the ODE  $y' = f(x, y)$  is guaranteed a solution exists. Furthermore since  $\frac{\partial f}{\partial y} = \frac{x}{y}$  is continuous everywhere except when  $y = 0$ , we're also guaranteed a unique solution for a while.

28) Since  $f(x, y) = \frac{x-1}{y}$  is continuous on the two rectangles given by  $\{-\infty < x < \infty, 0 < y\}$  and  $\{-\infty < x < \infty, y < 0\}$ , but the initial condition  $y(1) = 0$  falls between these rectangles, the ODE  $y' = f(x, y)$  is not guaranteed a solution exists.

32)  $y(x) = 0$  is a solution for all  $x$ -values. We can solve this ODE by separation of variables to get the solution  $y(x) = x^3$  (as discussed in class). Since the  $\frac{\partial}{\partial y} 3y^{2/3} = 2y^{-1/3}$  is not continuous at  $y = 0$ , the theorem does not guarantee us uniqueness. Thus, the two solutions do not conflict with the theorem.

CPA a) Plugging in  $y = ax + b$  into the ODE  $y' = \sin(x - y)$  yields the equation

$$a = \sin((1 - a)x + b).$$

Since the left hand side is constant, the right hand side must also be constant. This is only true when  $(1 - a) = 0$  or  $a = 1$ . This leaves us with the equation

$$1 = \sin(b)$$

to be solved for  $b$ . This yields  $b = n\pi/2$  for  $n = \dots, -2, -1, 0, 1, 2, \dots$

b) No, we would need  $C \rightarrow \infty$  for the initial condition  $y(\pi/2) = 0$  to satisfy  $y(x) = x - 2 \tan^{-1} \left( \frac{x-2-C}{x-C} \right)$ .

CPB Similar arguments to CPA.

1.4 2) Separating the variables of the ODE  $y' + 2xy^2 = 0$  yields the equation

$$\frac{1}{y^2} dy = -2x dx$$

which we integrate up to get

$$-\frac{1}{y} = -x^2 + C.$$

Solving for  $y$  explicitly we get  $y = \frac{1}{x^2 + C}$  ( $C$  has gone to  $-C$  because it is a general constant).

12) Separating the variables of the ODE  $yy' = x(y^2 + 1)$  yields the equation

$$\frac{y}{y^2 + 1} dy = x dx$$

which we integrate up (using  $u$ -substitution) to get

$$\frac{1}{2} \ln(y^2 + 1) = \frac{1}{2} x^2 + C.$$

Solving for  $y^2$  we get the implicit general solution  $y^2 = Ce^{x^2} - 1$ .

20) Separating the variables of the ODE  $y' = 3x^2(y^2 + 1)$  yields the equation

$$\frac{1}{y^2 + 1} dy = 3x^2 dx$$

which we integrate up (using  $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}$ ) to get

$$\tan^{-1}(y) = x^3 + C.$$

Solving for  $y$  we get the explicit general solution  $y = \tan(x^3 + C)$ . Using the initial condition  $y(0) = 1$ , we get that  $C = \tan^{-1}(1) = \pi/4$ . The particular solution is then  $y(x) = \tan(x^3 + \pi/4)$ .

26) Separating the variables of the ODE  $y' = 2xy^2 + 3x^2y^2$  yields the equation

$$\frac{1}{y^2} dy = 2x + 3x^2 dx$$

which we integrate up to get

$$-\frac{1}{y} = x^2 + x^3 + C.$$

Solving for  $y$  we get the explicit general solution  $y = -\frac{1}{x^2 + x^3 + C}$ . Using the initial condition  $y(1) = -1$ , we get that  $1 = \frac{1}{2+C}$ , so that  $C = -1$ . The particular solution is then  $y(x) = -\frac{1}{x^2 + x^3 - 1}$ .

42) Given the initial value problem for barometric pressure in terms of altitude,  $x$ ,  $p' = -0.2p$  with initial condition  $p(0) = 29.92$ , we are asked to find the pressure at 10,000ft and at 30,000ft. We separate variables to solve for  $p$  and get that  $p(x) = 29.92e^{-0.2x}$ . Plugging in our two altitudes gives us  $p(10,000ft = 1.9mi) = 29.92e^{-0.2 \cdot 1.9} \approx 20.5$  and  $p(30,000ft = 5.7mi) = 29.92e^{-0.2 \cdot 5.7} \approx 9.6$ . If people cannot survive at  $p < 15$  then solving for  $x$  at this  $p$ -value tells us the altitude at which this occurs (should be between 10,000ft and 30,000ft from our previous calculation). Solving for  $x$  in  $15 = 29.92e^{-0.2x}$  yields  $x \approx 3.45mi \approx 18,200ft$ .

CP Consider the ODE

$$\frac{dx}{dt} = \frac{m}{n}x - \frac{1}{n}x^2.$$

Letting  $m = 9$  and  $n = 7$ , we get the ODE

$$\frac{dx}{dt} = \frac{1}{7}(9x - x^2).$$

As  $t \rightarrow \infty$  it appears that the population approaches 9.

b) Maple gives the general solution as  $x(t) = 9\frac{1}{1+9Ce^{-9/7t}}$ . As  $t \rightarrow \infty$ ,  $x \rightarrow 9$  as expected.

c) We ask the question how long does it take for a population to grow by 80%? Letting  $x(0) = x_0$  we get the particular population  $x(t) = \frac{9x_0}{x_0 + (9-x_0)e^{-9/7t}}$ . The population is  $x = 0.8x_0$  when  $t = -7/9 \ln(\frac{0.8x_0}{9-x_0})$ . We can now plug in whatever initial population we like to get a specific  $t$ .

