Math 5470/6440 Final
Due Tuesday, December 11 at 3pm

Name:

1. Two-dimensional non-linear system.

a) Perform the phase-plane analysis of the following system. Include linear stability analysis of the fixed points, nullclines, vector field, representative trajectories

$$\dot{x} = y - x^3 + x$$
$$\dot{y} = \frac{1}{2}x - y$$

b) Introduce a parameter A in the original system such that changing A would produce a saddle-node bifurcation.
c) Introduce a parameter B in the original system such that changing B would produce a pitchfork bifurcation.

2.

$$\dot{x} = -y + \mu x - x + xy^2,$$
$$\dot{y} = x + \mu y - y - x^2.$$

a) Find eigenvalues of the Jacobian at origin.
b) Looking at eignevalues only, show that there is a Hopf bifurcation at $\mu = 1$
c) PLot phase portrait near the orgin for μ just below and just above $\mu = 1$ (assume that the bifurcation is supercritical).
d) Sketch a piece of bifurcation diagram near $\mu = 1$ and near $(x, y) = (0, 0)$.

3. A toy model for forest growth is given by

$$\dot{H} = -H(H - \frac{1}{2})(H - 1) - \mu, \quad \mu \geq 0,$$

where H is a measure of forest's health (1=healthy, 0=almost gone), μ is the amount of logging.

a) Plot all qualitatively different vector fields that are encountered as $\mu \geq 0$ varies. INdicate stability of fixed points.
b) Identify bifurcations you have encountered in a). Estimate the critical value of μ (if you can estimate it from looking at graphs, without explicit computation, that will suffice, but do explain what information from the graphs you needed to look up).
c) Start with the forest with $H(0) = 0.6$. Let it grow ($\mu = 0$) for a long time (to the equilibrium), then start logging ($\mu = 10$) for a long time, then stop logging again ($\mu = 0$). Plot the time course of the forest's health throughout these actions. Will the forest ever recover to its original state?

4. Consider

$$\dot{x} = -x + ay + x^2 y,$$
$$\dot{y} = b - ay - x^2 y.$$

Show that for appropriate values of $a, b > 0$ this system has a periodic solution.

5. Consider the cubic map $x_{n+1} = f(x_n)$, where $f(x) = rx - x^3$.

a) Find the fixed points and analyze their stability.
b) To find 2-cycles, suppose that $f(p) = q$ and $f(q) = p$. Show that p, q are roots of the equation $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0$ and use this to find all the 2-cycles.

6. Draw your own self-similar fractal (give a rule how to build it and draw several initial iteration) similar to one of those we considered in class. Determine the similarity dimension of your fractal.