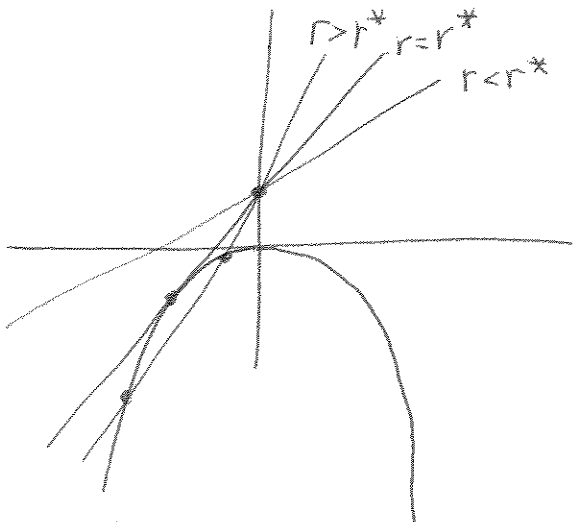
3.1.1.

$$\dot{x} = 1 + rx + x^2$$

(I)



$$f(x) = 0 \iff f_1(x) = f_2(x)$$

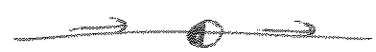
$$f_1(x) = 1 + rx$$
$$f_2(x) = -x^2$$

(II) The picture is symmetric w.r.t. r.
So, if we take $r^* > 0$, then there are 2 fixed points for $r > r^*$ and $r < -r^*$, no f. points $-r^* < r < r^*$; bifurcations at $r = \pm r^*$.

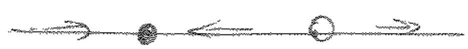
(III) $r > 0$.

1. $r < r^*$

2. $r = r^*$



3. $r > r^*$



$r < 0$ – same (symmetric)

(IV) Determine r^*:

$$f_1(x^*; r^*) = f_2(x^*; r^*)$$
$$f_1'(x^*; r^*) = f_2'(x^*; r^*)$$

$$\begin{cases} 1 + rx = -x^2 \\ r = -2x \end{cases}$$

$$1 - 2x^2 = -x^2$$
$$1 = x^2$$
$$x^* = \pm 1$$
$$r^* = \mp 2$$

By our convention $r^* = 2$
Bifurcations at ± 2.

(V) This is a saddle-node bifurcation. Either see it graphically from I, III, or Taylor expansion near $x = -1$, $r = 2$.

$$f(x) \approx f(x^*, r^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*, r^*} (x - x^*) + \left.\frac{\partial f}{\partial r}\right|_{x^*, r^*} (r - r^*) +$$

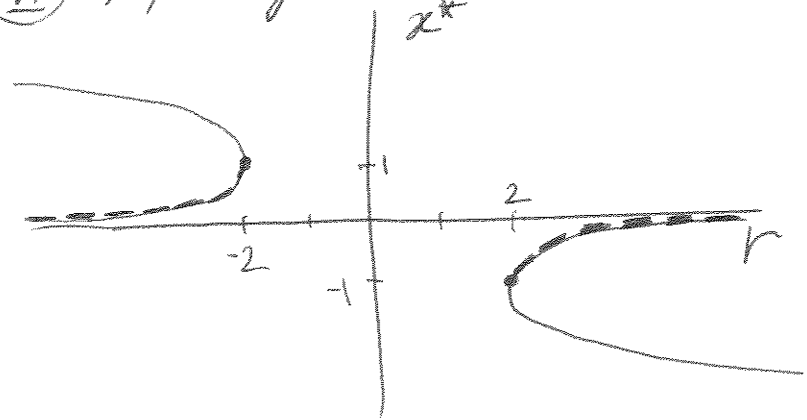
$$+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x^*, r^*} (x - x^*)^2 + \ldots$$

$$f(x) = 0 + 0 \cdot (x - x^*) + (-1) \cdot (r-2) + \frac{1}{2}\, 2\, (x+1)^2 + O(x^3, r^2)$$

$$= (-1)(r-2) + (x+1)^2 = \tilde{r} + \tilde{x}^2$$ – normal form for s.-n. bifurcation

$$(\tilde{r} = (-1)(r-2),\ \tilde{x} = x+1)$$

VI Bif. diagram



4.3.5

$$\dot{\theta} = \mu + \cos\theta + \cos 2\theta =$$
$$= \mu + \cos\theta + \cos^2\theta - \sin^2\theta = \mu + \cos\theta + 2\cos^2\theta - 1 =$$
$$= 2\cos^2\theta + \cos\theta + (\mu - 1)$$

$$f(\theta) = 0 \iff 2y^2 + y + (\mu-1) = 0 \quad (y = \cos\theta)$$
$$D = 1 - 8(\mu-1) = 9 - 8\mu$$

$9 - 8\mu < 0$

$\mu > \frac{9}{8}$

no solutions

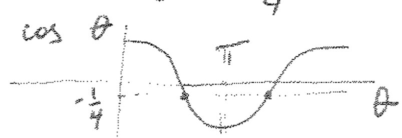
no fixed points

$9 - 8\mu = 0$

$\mu = \frac{9}{8}$

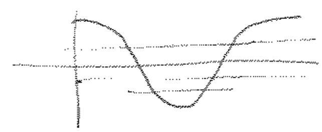
$y = \frac{-1}{4}$

$\cos\theta = -\frac{1}{4}$



$9 - 8\mu > 0 \quad \mu < \frac{9}{8}$

$y = \frac{-1 \pm \sqrt{9-8\mu}}{4}$



$-\frac{1}{4} - \frac{\sqrt{9-8\mu}}{4} = -1$

$1 + \sqrt{9-8\mu} = 4$

$9 - 8\mu = 9$

$\mu = 0$

$\cos\theta = \frac{-1-\sqrt{9-8\mu}}{4}$

has 2 solutions

when $0 < \mu \le \frac{9}{8}$

$-\frac{1}{4} + \frac{\sqrt{9-8\mu}}{4} = $ [illegible]

$\sqrt{9-8\mu} = 5$

$9 - 8\mu = 25$

$8\mu = 16$

$\mu = 2$

$\cos\theta = \frac{-1+\sqrt{9-[illegible]}}{4}$

has 2 solu[illegible]

when $-2 < \mu \le$ [illegible]

$\mu > \frac{9}{8}$

$\mu = \frac{9}{8}$

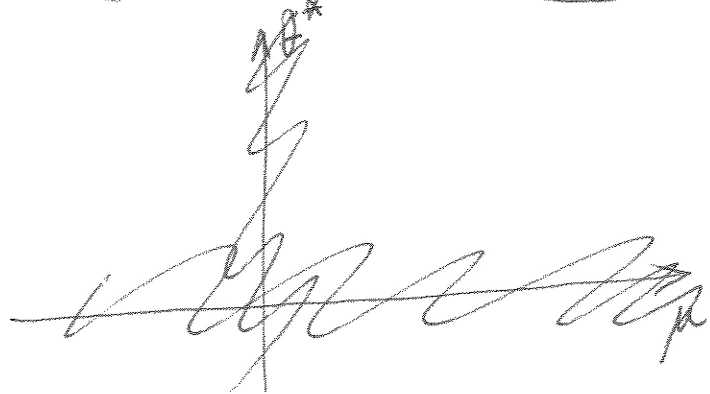
$0 < \mu < \frac{9}{8}$

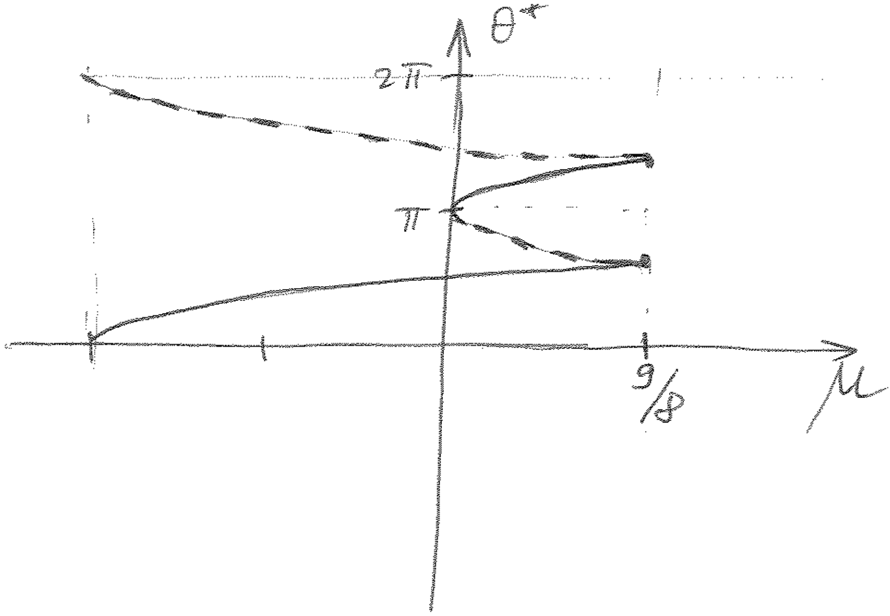
$\mu = 0$

$-2 < \mu < 0$

$\mu = -2$

$\mu < -2$





θ*
2π
π
9/8
μ

5.2.8

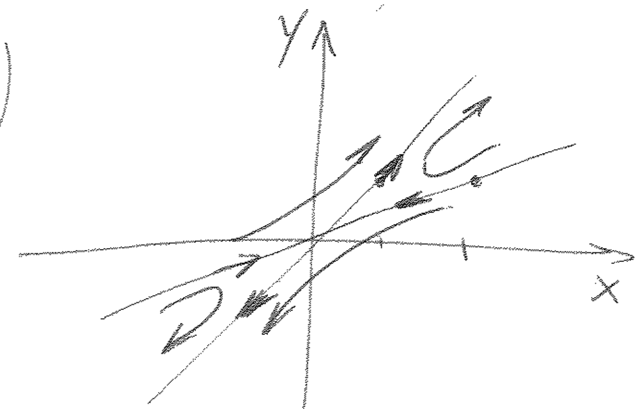
$\dot{x} = -3x + 4y$ $\quad A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$ $\quad \tau = 0$ $\quad$ saddle

$\dot{y} = -2x + 3y$ $\quad \Delta = -1$

or $\quad -9 + \lambda^2 + 8 = 0 \quad \lambda = \pm 1$ saddle

$1: \begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$-1: \begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



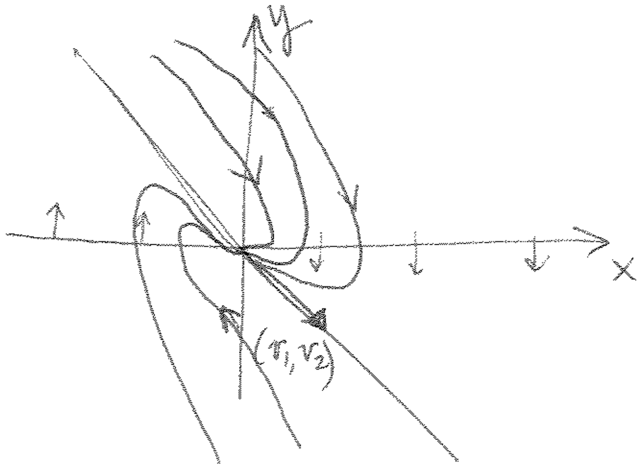
5.2.10

$\dot{x} = y$
$\dot{y} = -x - 2y$

$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ $\left| \begin{matrix} -\lambda & 1 \\ -1 & -2-\lambda \end{matrix} \right| = \lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1$. stable

$\begin{pmatrix} +1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ degenerate node



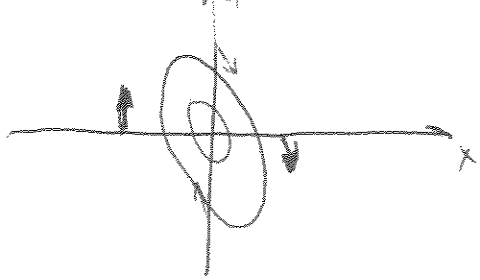
5.2.7

$\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$ $A = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix}$

$\det \begin{pmatrix} 5-\lambda & 2 \\ -17 & -5-\lambda \end{pmatrix} = \lambda^2 - 25 + 34 = \lambda^2 + 9 = 0$

$\lambda = \pm 3i$ center

or $\left. \begin{matrix} \tau = 5 - 5 = 0 \\ \Delta = 9 > 0 \end{matrix} \right\}$ center



6.3.2

$\cancel{\dot{x} = \sin y}$

$$\dot{x} = \sin y$$
$$\dot{y} = x - x^3$$

fixed points $\sin y = 0$ $\quad y = \pi n,\ n \in \mathbb{Z}$

$x - x^3 = 0$ $\quad x = 0, \pm 1.$

Linearization $\begin{pmatrix} 0 & \cos y \\ 1-3x^2 & 0 \end{pmatrix}$

at $\underline{y = 2\pi n, \quad x = 0}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \lambda = \pm 1$ $\quad$ <u>saddle</u>

$y = \pi + 2\pi n, \quad x = 0$

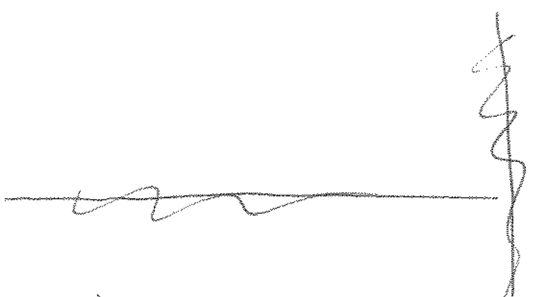
$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \lambda = \pm i$ $\quad$ <u>center</u>

$y = 2\pi n, \quad x = \pm 1$

$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ $\quad \lambda = \pm i\sqrt{2}$ $\quad$ <u>center</u>

$y = \pi + 2\pi n, \quad x = \pm 1$

$A = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$ $\quad \lambda = \pm\sqrt{2}$ $\quad$ <u>saddles</u>

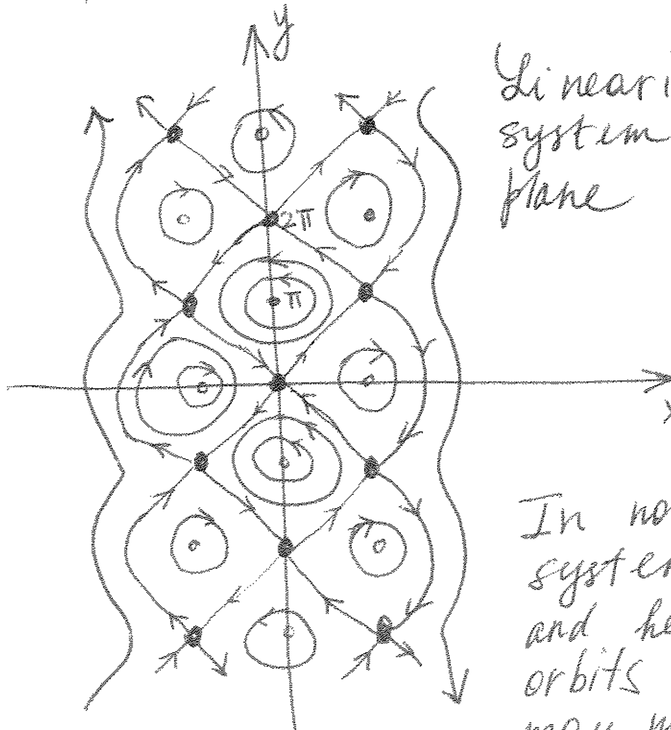


at $(0,0)$ eigenvectors

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ $\quad v_1 = 1,\ v_2 = +1$ $\quad \lambda = 1$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ $\quad v_1 = 1,\ v_2 = -1$ $\quad \lambda = -1$

All other direction arrows follow.



Linearized system phase plane

In non-linear system centers and heteroclinic orbits may or may not persist

6.3.6

$$\dot{x} = xy - 1$$
$$\dot{y} = x - y^3$$

f. points: $\begin{cases} xy = 1 \\ x = y^3 \end{cases}$ $\quad y^4 = 1 \quad y = \pm 1 \quad x = \pm 1 \quad (1,1)\ (-1,-1)$

Linearization $\begin{pmatrix} y & x \\ 1 & -3y^2 \end{pmatrix}$

$(1,1)$: $A = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$ $\quad \begin{vmatrix} 1-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0$

$$\lambda^2 - \lambda + 3\lambda - 3 - 1 = 0$$
$$\lambda^2 + 2\lambda - 4 = 0$$
$$\lambda = -1 \pm \sqrt{5}$$

saddle

$(-1,-1)$: $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$ $\quad \lambda^2 + 4\lambda + 4 = 0$

$$\lambda = -2 \pm 0$$

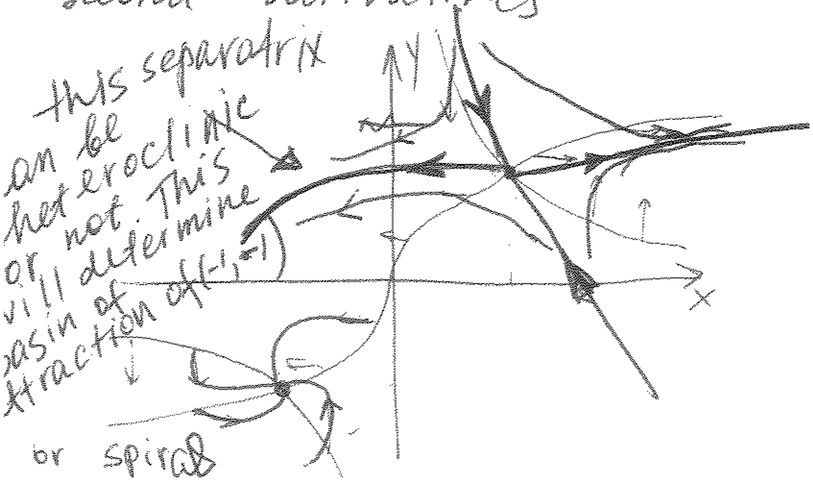
$$\det \begin{pmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{pmatrix} = \lambda^2 + 3\lambda + \lambda + 3 + 1$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ – one eigenvector

stable degenerate node

In the non linear system $(-1,-1)$ can be either a stable spiral or a stable node. Possible way to distinguish – numerically, or by looking at the second derivatives



Nullclines $y = \frac{1}{x}$

$x = y^3$

7.3.1

$$\dot{x} = x - y - x(x^2 + 5y^2)$$
$$\dot{y} = x + y - y(x^2 + y^2)$$

a) at $(x,y) = (0,0)$ linearization is

$$\dot{x} = x - y \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad (1-\lambda)^2 = -1$$
$$\dot{y} = x + y \qquad\qquad\qquad (1-\lambda) = \mp i$$

$\lambda = 1 \pm i \quad Re(\lambda) > 0$

unstable spiral

b) $r\dot{r} = x\dot{x} + y\dot{y}$, $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$

$$r\dot{r} = x(x - y - x(x^2 + 5y^2)) + y \cdot (x + y - y(x^2 + y^2)) =$$
$$= x^2 - xy - x^2(x^2 + 5y^2) + xy + y^2 - y^2(x^2 + y^2) =$$
$$= r^2 - (x^2 + y^2)(x^2 + y^2) - 4x^2y^2 = r^2 - r^4 - r^4 \sin^2(2\theta)$$
$$\dot{r} = r(1 - r^2 - r^2 \sin^2(2\theta))$$
$$\dot{\theta} = (x^2 + yx - yx(x^2 + y^2) - xy + y^2 + xy(x^2 + 5y^2)))/r^2 =$$
$$= (r^2 + xy \cdot 4y^2)/r^2 = 1 + 2r^2 \sin(2\theta)\sin^2\theta$$

c) r_1 such that at $r = r_1$ $\dot{r} > 0$

$$\dot{r} = r_1(1 - r_1^2 - r^2 \sin^2(2\theta)) > 0$$
$$r^2(1 + \sin^2(2\theta)) < 1$$
$$1 + \sin^2(2\theta) < \frac{1}{r^2} \qquad \frac{1}{r^2} > 2 \qquad r^2 < \frac{1}{2} \qquad r < \frac{1}{\sqrt{2}}$$

d) r_2 such that at $r = r_2$ $\dot{r} < 0$

$$1 + \sin^2(2\theta) > \frac{1}{r^2}$$
$$1 > \frac{1}{r^2} \qquad r > 1$$

e) No fixed points inside ring $r_1 < r < r_2$, so Poincaré Bendixon thm.

7.3.3

$\dot{x} = x - y - x^3$

$\dot{y} = x + y - y^3$

1. Using the same method as in 7.3.1

$$\dot{r} = r\left(1 - r^2 + \tfrac{1}{2} r^2 \sin^2(2\theta)\right)$$

$r_1 \leq 1$, $r_2 \geq \sqrt{2}$ $\qquad r_1 < r < r_2$ – trapping region

2. Graphically $(0,0)$ is the only equilibrium, so by P.-B. theorem there is a cycle.

7.3.5

Same method as in 7.3.1 yields

$$\dot{r} = r\left(-1 + r^2(1 + \sin^2\theta)\right)$$

when $r < \frac{1}{\sqrt{2}}$ $\quad \dot{r} < 0$

$r > 1$ $\quad \dot{r} > 0$

Apply P.-B. theorem to a system with time reversed. This system has a cycle. A cycle in a reversed-time system is also ~~Original~~ a cycle in the original system.

7.5.3.

$$\ddot{x} + k(x^2-4)\dot{x} + x = 1 \quad \text{for } k \gg 1$$

· Following example 7.5.1 – 7.5.2

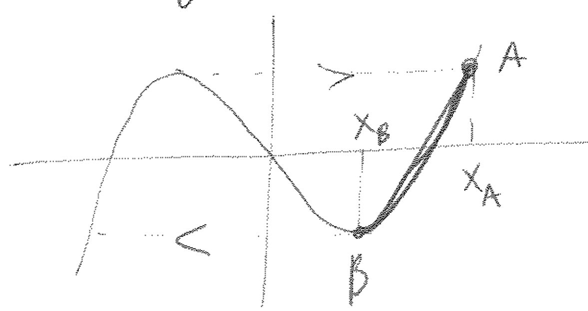
$$F(x) = \frac{1}{3}x^3 - 4x$$

$$\omega = \dot{x} + kF(x) \qquad y = \frac{\omega}{k}$$

$$\dot{\omega} = 1 - x$$
$$\dot{x} = \omega - kF(x)$$

$$\boxed{\begin{aligned} \dot{y} &= \frac{1}{k}(1-x) \\ \dot{x} &= k(y - F(x)) \end{aligned}}$$

x - fast x goes to $y = F(x)$
y - slow



$$T = 2\int_{t_A}^{t_B} dt$$

$y \approx F(x) \Rightarrow \frac{dy}{dt} = F'(x) \cdot \frac{dx}{dt} = (x^2-4) \cdot \frac{dx}{dt}$. On the other hand $\frac{dy}{dt} = \frac{1}{k}(1-x)$

$$\frac{1}{k}(1-x) = (x^2-4)\frac{dx}{dt}$$

$$dt = k\frac{(x^2-4)}{1-x}dx = k \cdot \left[-(x+1) - \frac{3}{1-x}\right]dx.$$

$x_B : F'(x_B) = 0, \ x_B = 2$

x_A: $F(x_A) = F(-x_B)$.

$x_A = 4$.

$$T = 2\int_{t_A}^{t_B} dt \approx 2\int_{x_A}^{x_B} k \cdot \left[-(x+1) + \frac{3}{-1+x}\right]dx =$$

$$= 2k\left[\frac{(x+1)^2}{2} + 3\ln(-1+x)\right]\Bigg|_2^4 = 2k\left[\frac{25}{2} - \frac{9}{2} + 3(\ln 3 - \ln 1)\right] = k(16 + 3\ln 3)$$

7.5.4

$$\ddot{x} + \mu f(x)\dot{x} + x = 0$$

$f(x) = -1$ for $|x|<1$; $f(x) = 1$ for $|x|>1$.

a) Same change of variables as in Ex. 7.5.1,

$$\dot{y} = -\frac{x}{\mu}$$

$$\dot{x} = \mu\left(y - F(x)\right), \text{ where } F'(x) = f(x)$$

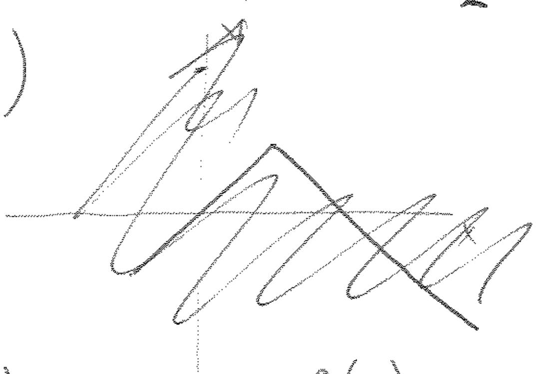
$F(x)$ - continuous

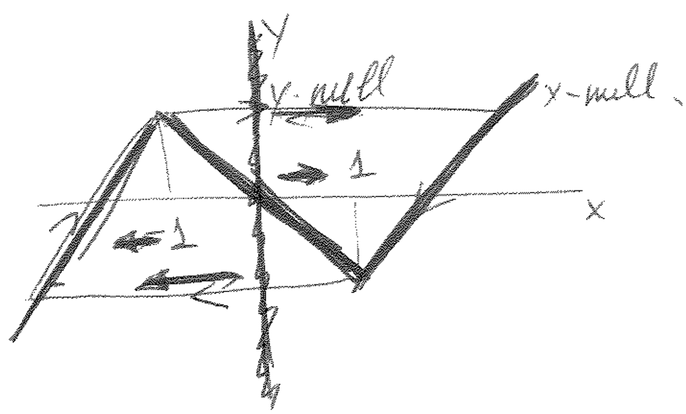
$$F(x) = \begin{cases} -x + c_1, & |x| \le 1 \\ x + c_2, & x > 1 \\ x + c_3, & x < -1 \end{cases}$$

at $x = 1$ $\quad -1 + c_1 = 1 + c_2 \Rightarrow c_2 = -2 + c_1$

$x = -1$ $\quad +1 + c_1 = -1 + c_3 \Rightarrow c_3 = +2 + c_1$. Take $c_1 = 0$.

b)



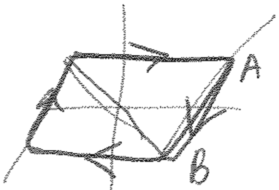


c) $\mu \gg 1 \Rightarrow |\dot{x}| \sim O(\mu) \gg 1$

$|\dot{y}| \sim O(\mu^{-1}) \ll 1$ away from $y = F(x)$ motion is horizonta

Arrows shown in the plane.

Cycle



d) $T = 2\int_{t_A}^{t_B} dt \approx 2\int_{x_A}^{x_B} -\frac{\mu}{x} f(x)\,dx = 2\int_1^3 \frac{1}{x}\,dx = 2\ln$

$$-\frac{x}{\mu} = \frac{dy}{dt} = f(x)\cdot\frac{dx}{dt} \qquad dt = -\frac{\mu}{x}\cdot f(x)\,dx$$

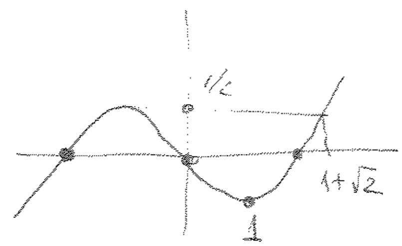
7.5.5

$$\ddot{x} + \mu(|x|-1)\dot{x} + x = 0$$

$$\dot{x} = \mu(y - F(x))$$
$$\dot{y} = -\frac{x}{\mu}$$

$$F(x) = \begin{cases} \frac{x^2}{2} - x + c_1, & x \geq 0 \\ -\frac{x^2}{2} - x + c_2, & x \leq 0 \end{cases}$$

$$F(0) = c_1 = c_2 = 0$$



$$T = 2\int_{t_A}^{t_B} dt = 2\int_{x_A}^{x_B} -\frac{\mu}{x}(|x|-1)\,dx =$$

$$= 2\int_{1}^{1+\sqrt{2}} \mu \cdot \left(1 - \frac{1}{x}\right) dx = 2\mu\left(\sqrt{2} - \ln(1+\sqrt{2})\right)$$