Math 5110/6830
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Solutions for practice test 2

1. (a) Immune cells destroy step.

(b) Reproduction rate of strep is roughly linear for small number of bacteria and saturates for higher number of bacteria. Per capita reproduction goes to zero as the number of bacteria increases. The reason for this is that as bacteria run out of space the growth of population slows down

(c) Cytokines are generated when an immune cell encounters a bacterium. This is the signal that your immune cells (and other cells too) secrete to alert other immune cells that there is a foreign invader. They are degraded naturally at rate μ

(d) For fast dynamics of C we need its production and degradation to be fast, i.e. the rate of change of the amount of C to be large, i.e. the derivaitive to be large, i.e. γ and μ need to be large. Then we can assume that, say, $1/\gamma$ is small, i.e.

$$0 = \frac{1}{\gamma}\frac{dC}{dt} = IS - \frac{\mu}{\gamma}C$$

$$C = \frac{\gamma IS}{\mu}$$

Plugging this into the other two equations so that we have a system of 2 ODEs:

$$\frac{dS}{dt} = \frac{\rho S}{k_1 + S} - \alpha IS$$

$$\frac{dI}{dt} = \frac{\beta(\gamma IS)^2}{(k_2\mu)^2 + (\gamma IS)^2} - \alpha pIS - \delta I$$

2. (a) There are possibly three equilibria points. The easiest one to write down is $(0,0)$. The other two are easier to see from the phase plane (but can also be found analytically). S-nullcline has two parts:

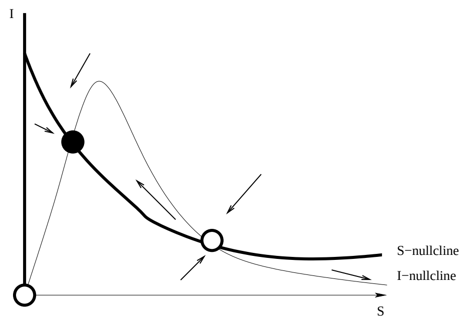
$$S = 0$$

$$I = \frac{\rho}{\alpha(k_1 + S)}$$

I-nullcline:

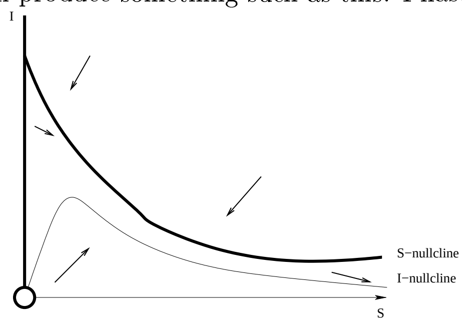
$$I = \frac{\beta S}{(k_2 + S)(\alpha pS + \delta)}$$

Phase-plane diagram:

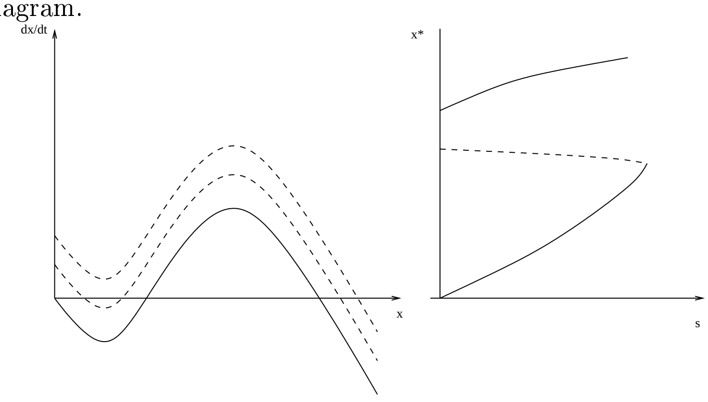


The (0,0) point is unstable. The first non-zero point is stable, and the other point is unstable. With a bit of algebra you can check this analytically.

(b) In this case, the only equilibria point is (0, 0) which is unstable. The strep win! A fast growing bacteria coupled with a slow moving immune system will produce something such as this. Phase plane diagram:



3. (a-c) The curve for $s = 0$ is solid, $s > 0$ dashed. On the right is bif. diagram.



(d) When s is slowly increased, $x(\tau)$ will approach the lower stable equilibria point. However, once the value of s gets large enough, we lose two of the fixed points. So, $x\tau)$ will then move towards the upper stable point.

(f) If it has been allowed to stabilize at the upper fixed point, then, it will stay there. With $s = 0$, there are again 3 fixed points but the middle one is unstable so trajectories will move away from it.

4. (a) $\frac{dP}{dt}$ represents the rate of change of the performance over time, ie. "how fast" someone picks up a new skill.

(b) When $M \geq P$, $\frac{dP}{dt} \geq 0$, so $P(t)$ is increasing or staying constant in time. When $M < P$, $\frac{dP}{dt} < 0$ which means that $P(t)$ is decreasing in time. We expect that with more and more training, a person will never have a decrease in performance. Notice that if we start with P below M, P can never get larger than M. If $P = M$, P will remain constant. This model is reasonable. We interpret M as the level when someone has mastered the skill, and k as the rate of skill acquisition.

(c) A reasonable initial condition would be $P(0) = 0$, ie. no previous knowledge.

(d) Note that the equilibria point is $P^* = M$. It is stable for k positive, and unstable otherwise.

(e) The bifurcation occurs at $k = 0$. It's a transcritical bifurcation.

5. (a)

$$\frac{dX}{dt} = -k_1 AX + 2k_1 AX - k_2 XY = k_1 AX - k_2 XY$$

$$\frac{dY}{dt} = k_2 XY - k_3 Y$$

(B) Equilibria points are (0,0) and $(\frac{k_3}{k_2}, \frac{k_1}{k_2} A)$

$$J(X^*, Y^*) = \begin{pmatrix} k_1 A - k_2 Y^* & -k_2 X^* \\ k_2 Y^* & k_2 X^* - k_3 \end{pmatrix}$$

For (0,0) we have

$$J(x^*, Y^*) = \begin{pmatrix} k_1 A & 0 \\ 0 & k_3 \end{pmatrix}$$

so $\lambda_1 = k_1 A$ and $\lambda_2 = -k_3$ and (0,0) is a saddle.
At the other point

$$J\left(\frac{k_3}{k_2}, \frac{k_1}{k_2 A}\right) = \begin{pmatrix} 0 & -k_3 \\ k_1 A & 0 \end{pmatrix}$$

Eigenvalues are imaginary with zero real parts. It is a center in the linearized system. In the full nonlinear system it can becaome a stable or unstable spiral