Math 5110/6830
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Practice for test 2

1. A team of modelers led by V. Gueraldi developed a three- dimensional system of ordinary differential equations to describe the dynamics of Streptococcus (strep) bacteria, S, immune cells that attack and destroy them, I, and a cytokine C (a chemical important in regulating the immune response).

$$\frac{dS}{dt} = \frac{\rho S}{k_1 + S} - \alpha IS,$$

$$\frac{dI}{dt} = \frac{\beta C^2}{k_2^2 + C^2} - \alpha pIS - \delta I,$$

$$\frac{dC}{dt} = \gamma IS - \mu C.$$

All parameters are assumed to be non-negative.

a. How do immune cells affect strep?

b. Describe the per capita reproduction of strep in the absence of immune cells, and give a reason why it might follow the given form.

c. When are cytokines generated? How are they degraded?

d. Suppose that cytokine dynamics are fast relative to other processes in the model. What parameters will be large? Use this to reduce this model to a two dimensional system.

2. N. K. Cole et al. derived a similar but subtly different two-dimensional model of the interaction between strep S and immune cells I given by

$$\frac{dS}{dt} = \frac{\rho S}{k_1 + S} - \alpha IS,$$

$$\frac{dI}{dt} = \frac{\beta S}{k_2 + S} - \alpha pIS - \delta I.$$

All parameters are assumed to be non-negative.

a. Find the equilibria and nullclines. Draw the phase plane, complete with direction arrows, in the case with the most possible equilibria. Find the stability of the equilibria graphically and of the zero euilibrium analytically.

b. Do the same in the case with the fewest equilibria. What would happen to the infection in the long run?

3. Consider a gene that is activated by the presence of a biochemical substance S. Let $g(t)$ denote the concentration of the gene product at time t, and

assume that the concentration of S, denoted by s_0, is fixed. A model describing dynamics of g is as follows:

$$\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

where the k's are positive constants.

a. This equation can be put in a dimensionless form:

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2},$$

where $r > 0$ and $s \geq 0$. Plot $\frac{dx}{dt}$ versus x for $s = 0$ and $r = 0.4$.

b. On the same axes as a. sketch the graph of $\frac{dx}{dt}$ versus x for various values of $s > 0$.

c. Make a qualitative sketch of the bifurcation diagram, showing the location and stability of the steady states x^* with s as the parameter. Identify any bifurcations.

d. Assume that initially there is no gene product, that is $x(0) = 0$, and suppose that s is slowly increased from zero (i.e. th biochemical substance S is slowly introduced). What happens t $x(\tau)$? Why

e. What happens if s goes back to zero after reaching some high value? Does the gene turn off again?

4. (dV 3.9.2) Psychologists and biologists interested in learning theory study learning curves. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t.

a)What does dP/dt represent?

b)Discuss why the differential equation

$$\frac{dP}{dt} = k(M - P),$$

where k and M are constants is a reasonable model for learning. You may want to include a graph of dP/dt vs. P as part of your discussion. What is the meaning of k and M?

c)What would be a reasonable initial condition for the model?

c)What would be a reasonable initial condition for the model?

d)Sketch all qualitatively different vector fields as the parameters are varied.

e)Are there any bifurcations? At what parameter value? Does this bifurcation belong to any of the classes we discussed in class?

5. The following chemical reaction mechanism was studied by Lotka:

$$A + X \to^{k_1} 2X$$

$$X + Y \to^{k_2} 2Y$$

$$Y \to^{k_3} B$$

Assume that A and B are kept at constant concentration. a)Write a set of equations for the concentrations of X and Y using the law of mass action. b)Study the steady states of the system and their stability. What can you say about oscillatory solutions?