5110/6370, Fall 2009
Practice for midterm 1.
Will be discussed in class on October 6
The midterm is on October 8

Open book and notes, no graphing calculators, no cell phones, etc.

Many people think that the Guinea worm is the worlds most disgusting parasite of humans. A person is infected by drinking water contaminated with infected water fleas. One year later, a 3 foot long worm emerges, very painfully and over a period of a few weeks, from the persons leg. To relieve the pain, the victim goes into the water, where the worm releases thousands of eggs, which quickly infect the water fleas to begin the cycle again. There is no immunity, and people can be infected year after year.

1. Let H_t represent the number of people who have worms emerging in year t, and F_t represent the number of infected fleas in year t. The function $h(H_t)$ gives the number of fleas in year t infected per infected person in year t, and $g(F_t)$ gives the fraction of people who will be infected in year $t+1$. The total population size is N. The associated model is then

$$F_{t+1} = h(H_t)H_t$$

$$H_{t+1} = g(F_t)N.$$

Suppose $h(H_t) = \alpha$ (a constant) and $g(F_t) = \frac{F_t}{F_t+k}$.
a. Write the one-dimensional discrete-time dynamical system giving H_{t+1} as a function of H_t.
b. Find the steady states of this model (Indicate parameter ranges where the steady states exist and make biological sense)
c. Determine stability of the steady states
d. Let $N = 100$, $k = 1000$. Draw the bifurcation diagram wih α as the parameter.
e. Sketch the solution with $H_0 = 50$, $N = 100$, $k = 1000$ for $\alpha = 1500$ and $\alpha = 5$. Compare the long-term behavior of solutions. Why does it make sense at these values of α?

2. Suppose now that the population of people can change, rather than being fixed at N. Denote this total population by N_t. People with emerging worms in year t (there are H_t of them) produce no offspring, while those

without emerging worms in year t (there are $N_t - H_t$ of them) produce r offspring each in year $t+1$.
a. Extend the model in problem 1 to include an equation for N_{t+1} in terms of N_t and H_t.
b. Find the equilibria
c. Find their stability.

3. Ignore the complexities of the previous problems. Assume there are two kinds of people: those who are careful and avoid drinking potentially contaminated water (with population C_t) and those who are not (with population U_t). Suppose that children copy the behavior of their parents, at least initially. Careful parents have m_c careful children, and uncareful parents have m_u uncareful children. A fraction σ of all people survive, no matter how old they are or how they behave (this includes newborn children).
Of those that survive, a fraction p_c of careful people remain careful (the rest become uncareful) and a fraction p_u of uncareful people remain uncareful (the rest become careful). Careful people have no chance of being infected (this is the key to finally eradicating this disease).
a. Write a matrix equation describing the populations of careful and uncareful people. It should be of the form

$$\begin{pmatrix} C_{t+1} \\ U_{t+1} \end{pmatrix} = \sigma M \begin{pmatrix} C_t \\ U_t \end{pmatrix}$$

for some matrix M that does not depend on σ.
b. Suppose $p_c = p_u = 0.5$, $m_u = 0.05$ and $m_c = 0.15$. In this case, the smaller eigenvalue of the matrix M is approximately 0.1. Find the other eigenvalue. Write a general form of the solution. For what values of σ will the population grow? Are there any conditions under which people survive and the disease disappears?
c. Suppose that p_u is a decreasing function of U_t (because a large population of infected people makes people more likely to switch). Can this lead to elimination of the disease?

4. A receptor on a cell membrane can be in one of two states: bound or unbound. Let B_t and U_t represent the fraction in each state at time t. Suppose an unbound receptor binds during one second with probability p

and remains unbound otherwise. A bound receptor unbinds with probability $q(B_t) = \frac{1-B_t}{2-B_t}$ and remains bound otherwise.
a. Write a discrete-time dynamical system for B_t, using the fact that $B_t + U_t = 1$.
b. Check that $B^* = \frac{2p}{1+p}$ is an equilibrium. Find the other equilibrium yourself.
c. Find the stability of the equilibria.
d. Draw a nice cobweb diagram of this system.
e. What bifurcations occur as a function of p?

5. A receptor on a cell membrane can be in one of two states, bound or unbound. Let B_t and U_t represent the number in each state at time t. An unbound receptor binds during one second with constant probability p and a bound receptor unbinds with constant probability q. In addition, each unbound receptor creates r new unbound receptors, and each bound receptor has a probability δ of being destroyed (unbound receptors last forever). Suppose that $q + \delta < 1$.
a. Write a matrix equation describing this population of receptors.
b. What is the condition on r for the number of receptors to increase?
c. Set all the parameters equal to 1/2. What fraction of receptors will be bound in the long run?

6. Consider the situation in problem 5 but assume that r or δ is can be functions of B_t or U_t.
a. Find a relationship between r and δ that must hold at a non-zero equilibrium (write the equations and eliminate B^* and U^*). Does this have anything to do with 5b?
b. Suppose that $p = q = \delta = 1/2$ (δ is a constant rather than a function) and that r is a function of U_t only. Find the Jacobian at the non-zero equilibrium (you can use the condition found in a to get a value for $r(U^*)$), and find a condition on the derivative of $r(U)$ that makes it unstable.