Math 5110/6830
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Practice for test 2

1. A team of modelers led by V. Gueraldi developed a three- dimensional system of ordinary differential equations to describe the dynamics of Streptococcus (strep) bacteria, S, immune cells that attack and destroy them, I, and a cytokine C (a chemical important in regulating the immune response).

$$\frac{dS}{dt} = \frac{\rho S}{k_1 + S} - \alpha IS,$$

$$\frac{dI}{dt} = \frac{\beta C^2}{k_2^2 + C^2} - \alpha p IS - \delta I,$$

$$\frac{dC}{dt} = \gamma IS - \mu C.$$

All parameters are assumed to be non-negative.

a. How do immune cells affect strep? Do they alter birth or death rates?

b. Describe the per capita reproduction of strep in the absence of immune cells, and give a reason why it might follow the given form.

c. What does that C^2 term mean and why might it be there? What does k_2 mean?

d. What does the parameter p mean? What is a reasonable range of values and why?

e. When are cytokines generated? How are they degraded?

f. Suppose that cytokine dynamics are fast relative to other processes in the model. What parameters will be large? Use this to reduce this model to a two dimensional system.

2. N. K. Cole et al. derived a similar but subtly different two-dimensional model of the interaction between strep S and immune cells I given by

$$\frac{dS}{dt} = \frac{\rho S}{k_1 + S} - \alpha IS,$$

$$\frac{dI}{dt} = \frac{\beta S}{k_2 + S} - \alpha p IS - \delta I.$$

All parameters are assumed to be non-negative.

a. What is different about this model?

b. Find the equilibria and nullclines. Draw the phase plane, complete with direction arrows, in the case with the most possible equilibria. Find the stability of the equilibria.

c. Do the same in the case with the fewest equilibria. What would happen to the infection in the long run? Explain why large values of some parameters and small values of others tend to create this case.

3. Consider a gene that is activated by the presence of a biochemical substance S. Let $g(t)$ denote the concentration of the gene product at time t, and assume that the concentration of S, denoted by s_0, is fixed. A model describing dynamics of g is as follows:

$$\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

where the k's are positive constants.

a. Interpret all three terms on the right-hand side of the equation (be sure to mention the meaning of k's).

b. This equation can be put in a dimensionless form:

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2},$$

where $r > 0$ and $s \geq 0$. Plot $\frac{dx}{dt}$ versus x for $s = 0$ and $r = 0.4$.

c. On the same axes as b. sketch the graph of $\frac{dx}{dt}$ versus x for various values of $s > 0$.

d. Make a qualitative sketch of the bifurcation diagram, showing the location and stability of the steady states x^* with s as the parameter. Identify any bifurcations.

e. Assume that initially there is no gene product, that is $x(0) = 0$, and suppose that s is slowly increased from zero (i.e. the biochemical substance S is slowly introduced). What happens to $x(\tau)$? Why?

f. What happens if s goes back to zero after reaching some high value? Does the gene turn off again? Why?