

MATH 5110
Midterm

Do all four problems (25 points each). Write readable answers on the test, but feel free to use or hand in additional paper if necessary. Open book and notes, no calculators, no cheating.

Many people think that the Guinea worm is the world's most disgusting parasite of humans. A person is infected by drinking water contaminated with infected water fleas. One year later, a 3 foot long worm emerges, very painfully and over a period of a few weeks, from the person's leg. To relieve the pain, the victim goes into the water, where the worm releases thousands of eggs, which quickly infect the water fleas to begin the cycle again. There is no immunity, and people can be infected year after year.

1. Let H_t represent the number of people who have worms emerging in year t , and F_t represent the number of infected fleas in year t . The function $h(H_t)$ gives the number of fleas in year t infected per infected person in year t , and $g(F_t)$ gives the fraction of people who will be infected in year $t + 1$. The total population size is N . The associated model is then

$$\begin{aligned}F_t &= h(H_t)H_t \\H_{t+1} &= g(F_t)N\end{aligned}$$

Suppose $h(H_t) = \alpha$ (a constant) and $g(F_t) = \frac{F_t}{F_t + k}$.

- a. Write the one-dimensional discrete-time dynamical system giving H_{t+1} as a function of H_t .
- b. Find the equilibria of this model and their stability.
- c. Using the same form of $g(F_t)$, write a model where $h(H_t)$ is a decreasing function. On a cobweb or other diagram, illustrate a possible bifurcation this model could exhibit.

2. Suppose now that the population of people can change, rather than being fixed at N . Denote this total population by N_t . People with emerging worms in year t (there are H_t of them) produce no offspring, while those without emerging worms in year t (there are $N_t - H_t$ of them) produce r offspring in year $t + 1$.
- a. Extend the model in problem 1 to include an equation for N_{t+1} in terms of N_t and H_t .
 - b. Find the equilibria
 - c. Find their stability.

3. Consider the population W of worms living in the water during the relatively brief period when they are emerging from people. Suppose that the number of worms in the water changes at a rate proportional to H_t . The number of infected fleas increases at a rate proportional to the product of the number of worms and the number of uninfected fleas. Assume that the total population of fleas is the constant value N_F , and that the duration of this period is T .
- Write and solve a differential equation for the number of worms.
 - Write and solve a differential equation for the number of fleas. It is possible to separate variables.
 - Use these dynamics to write a discrete-time dynamical system for the number of infected hosts. Does it match the form in problem 1?

4. Ignore the complexities of the previous problems. Assume there are two kinds of people: those who are careful and avoid drinking potentially contaminated water (with population C_t) and those who are not (with population U_t). Suppose that children copy the behavior of their parents, at least initially. Careful parents have m_c careful children, and uncaredful parents have m_u uncaredful children. A fraction σ of all people survive, no matter how old they are or how they behave (this includes newborn children).

Of those that survive, a fraction p_c of careful people remain careful (the rest become uncaredful) and a fraction p_u of uncaredful people remain uncaredful (the rest become careful). Careful people have no chance of being infected (this is the key to finally eradicating this disease).

- a. Write a matrix equation describing the populations of careful and uncaredful people. It should be of the form

$$\begin{pmatrix} C_{t+1} \\ U_{t+1} \end{pmatrix} = \sigma M \begin{pmatrix} C_t \\ U_t \end{pmatrix}$$

for some matrix M that does not depend on σ .

- b. Suppose $p_c = p_u = 0.5$, $m_u = 0.05$ and $m_c = 0.15$. In this case, the smaller eigenvalue of the matrix M is approximately 0.1. Find the other eigenvalue. For what values of σ will the population grow? Are there any conditions under which people survive and the disease disappears?
- c. Suppose that p_u is a decreasing function of U_t (because a large population of infected people makes people more likely to switch). Can this lead to elimination of the disease?