

NAME: \_\_\_\_\_

**MATH 5110**  
**Midterm**

Do all four problems (25 points each). Write readable answers on the test, but feel free to use or hand in additional paper if necessary. Open book and notes, no calculators, no cheating. You might want to recall that  $\ln(2) \approx 0.7$ ,  $\sqrt{2} \approx 1.4$ ,  $\ln(3) \approx 1.1$ ,  $e \approx 2.7$  and  $\pi \approx 3.1415926535$ .

1. A receptor on a cell membrane can be in one of two states: bound or unbound. Let  $B_t$  and  $U_t$  represent the fraction in each state at time  $t$ . Suppose an unbound receptor binds during one second with probability  $p$  and remains unbound otherwise. A bound receptor unbinds with probability  $q(B_t) = \frac{1 - B_t}{2 - B_t}$  and remains bound otherwise.
  - a. Write a discrete-time dynamical system for  $B_t$ , using the fact that  $B_t + U_t = 1$ .
  - b. Check that  $B^* = \frac{2p}{1+p}$  is an equilibrium. Find the other equilibrium yourself.
  - c. Find the stability of the equilibria.
  - d. Draw a nice cobweb diagram of this system.
  - e. What bifurcations occur as a function of  $p$ ?

2. Consider the equilibrium  $B^* = \frac{2p}{1+p}$  given in problem 1. Suppose that there are different types of receptors with different values of  $p$  following the probability density function  $f(p)$ .
- Write an expression for the fraction bound after averaging over all types of receptors.
  - Suppose  $f(p) = 1$  for  $0 \leq p \leq 1$ . Find the fraction that are bound. Is this fraction higher or lower than with  $p = 1/2$  for all receptors?
  - Suppose that a receptor can switch between different values of  $p$ . Sketch a model describing such a system, making sure to make clear when the switches occur.

3. A receptor on a cell membrane can be in one of two states, bound or unbound. Let  $B_t$  and  $U_t$  represent the **number** in each state at time  $t$ . An unbound receptor binds during one second with constant probability  $p$  and a bound receptor unbinds with constant probability  $q$ . In addition, each unbound receptor creates  $r$  new unbound receptors, and each bound receptor has a probability  $\delta$  of being destroyed (unbound receptors last forever). Suppose that  $q + \delta < 1$ .
- Write a matrix equation describing this population of receptors.
  - What is the condition on  $r$  for the number of receptors to increase?
  - Set all the parameters equal to  $1/2$ . What fraction of receptors will be bound in the long run?

4. Consider the situation in problem **3** but assume that  $r$  or  $\delta$  is can be functions of  $B_t$  or  $U_t$ .
- What sort of dependence of  $r$  on  $B_t$  and  $U_t$  or of  $\delta$  on  $B_t$  and  $U_t$  might lead to the existence of a non-zero equilibrium?
  - Find a relationship between  $r$  and  $\delta$  that must hold at a non-zero equilibrium (write the equations and eliminate  $B^*$  and  $U^*$ ). Does this have anything to do with **3b**?
  - Suppose that  $p = q = \delta = 1/2$  ( $\delta$  is a constant rather than a function) and that  $r$  is a function of  $U_t$  only. Find the Jacobian at the non-zero equilibrium (you can use the condition found in **b** to get a value for  $r(U^*)$ ), and find a condition on the derivative of  $r(U)$  that makes it unstable. What bifurcations do you expect this system to undergo?