

**Math 5110/6830**  
**Instructor: Alla Borisjuk**  
**Homework 7.1**  
**Due: October 31**

1. (St: ch3) For each of the following exercises do:
- (i) sketch all the qualitatively different vector fields that occur as  $\mu$  is varied.
  - (ii) Find the bifurcation value of  $\mu$  graphically.
  - (iii) Find the bifurcation value of  $\mu$  analytically, if possible.
  - (iv) Identify the bifurcation.
  - (v) Sketch the bifurcation diagram with  $\mu$  as a parameter.

**a)**  $\dot{x} = 1 + rx + x^2$

**b)**  $\dot{x} = x - rx(1 - x)$

**c)**  $\dot{x} = x(r - e^x)$

**d)**  $\dot{x} = r + \frac{1}{2}x - x/(1 + x)$

2. (dV 3.9.2) Psychologists and biologists interested in learning theory study learning curves. A learning curve is the graph of a function  $P(t)$ , the performance of someone learning a skill as a function of the training time  $t$ .

**a)** What does  $dP/dt$  represent?

**b)** Discuss why the differential equation

$$\frac{dP}{dt} = k(M - P),$$

where  $k$  and  $M$  are constants is a reasonable model for learning. You may want to include a graph of  $dP/dt$  vs.  $P$  as part of your discussion. What is the meaning of  $k$  and  $M$ ?

**c)** What would be a reasonable initial condition for the model?

**d)** Sketch all qualitatively different vector fields as the parameters are varied.

**e)** Are there any bifurcations? At what parameter value? Does this bifurcation belong to any of the classes we discussed in class?

3. The normal form of the saddle-node bifurcation in two dimensions is

$$\dot{x} = \mu - x^2,$$

$$\dot{y} = -y.$$

**a)** Sketch the phase portraits as  $\mu$  varies. Notice that along the  $x$ -axis the behavior is exactly the same as in the one-dimensional case, and in  $y$  direction the behavior is very simple (what is it?).

**b)** Draw the nullclines for this system for the same  $\mu$  values as the phase-portraits in a). What behavior of the nullclines corresponds to the saddle-node bifurcation? (This is a generic fact!)

4. (St: ch.8.1) The following system describes an evolution of a protein (with concentration  $x$ ) and the activity of a gene which codes for the protein (represented by  $y$  - the concentration of the corresponding messenger RNA):

$$\begin{aligned}\dot{x} &= -ax + y, \\ \dot{y} &= \frac{x^2}{1+x^2} - by,\end{aligned}$$

where  $a$  and  $b$  are positive constants.

**a)** Sketch carefully or plot in Matlab the nullclines of the system for small  $a$ . How many fixed points are there?

**b)** Say in words and illustrate in a figure (sketch or Matlab) what happens to the nullclines as  $a$  increases?

**c)** Based on your observations about nullclines, what bifurcation, do you think, happens as  $a$  is increased? Sketch the bifurcation diagram without indicating stability. (You don't have to find out all details about the fixed points, such as their stability - just observe the qualitative behavior of the nullclines).

**d) (Extra credit)** Describe in words the underlying biological situation. What is the interaction between the gene and the protein (do they inhibit or promote each other? what quantities of each one are needed to influence the other?) What is the meaning of  $a$  and  $b$ ? Can you suggest why this type of interaction may be biologically important?

**5. (Extra credit)** Show that the first-order system  $\dot{x} = r - x - e^{-x}$  undergoes a saddle-node bifurcation as  $r$  is varied, and find the bifurcation value of  $r$ .