

Homework 6 Problems

1. For the plague model we saw in class:

$$\begin{aligned}\frac{dS}{dt} &= -\beta IS \\ \frac{dI}{dt} &= \beta IS - kI\end{aligned}$$

we noticed that there was a solution $S > 0$ that existed (this was our stable line). To prove this analytically:

- Make an equation $\frac{dS}{dI}$ and solve using separation of variables.
 - With the initial conditions $S(0) = S_0$ & $I(0) = 0$ and the terminal conditions $S(\infty) = S_\infty$ & $I(\infty) = 0$, write an algebraic equation for S_∞ . Show that $S_\infty = S_0$ is a solution.
 - Graph the equation for S_∞ and show that there is another solution. What are the conditions for this solution to exist? (HINT: let $f(S) = -S + \frac{k}{\beta} \ln(S)$ and find the maximum. What happens if you start above this max value? Below it?)
2. We have seen the disease model (the plague model with births):

$$\begin{aligned}\frac{dS}{dt} &= bS - \beta IS \\ \frac{dI}{dt} &= \beta IS - kI\end{aligned}$$

- Suppose first that diseased individuals recover at rate γ and return back to being susceptible. Analyze this system, including finding equilibria, their stability, and drawing a phase plane diagram.
 - Suppose instead that diseased individuals can also reproduce and produce uninfected offspring, but do so at a reduced rate (multiply b by a value less than 1). Analyze everything as before.
3. An unfortunate ecological interaction that has been recorded for over 1500 species in the animal kingdom is cannibalism. One model describing this is

$$\begin{aligned}\frac{dJ}{dt} &= r(J)A - \mu AJ - \theta J \\ \frac{dA}{dt} &= \theta J - \delta A\end{aligned}$$

Suppose that $r(J)$ is linearly increasing function of J : $r(J) = r_0 + \eta\mu J$ with $\eta < 1$.

- Explain every equation.
- Draw the phase-plane diagram.
- Find the equilibria and their stability.
- Under what conditions will this population survive?