

Math 5110/6830
Instructor: Alla Borisjuk
Homework 2.2
Due: September 12

1. (dV 2.4.6) Verify the results, illustrated in class, for the second-iterate map $f^2(x)$, where $f(x)$ is the logistic map $f(x) = r(1 - x)x$. Namely, do the following:

a) Compute $f^2(x)$.

b) Find the fixed points of $f^2(x)$. Verify that the non-trivial 2-cycle only exists for $r > 3$.

c) Compute $\frac{d}{dx}f^2(x)$.

d) Verify that the non-trivial 2-cycle is stable for $3 < r < 1 + \sqrt{6}$, and unstable for $r > 1 + \sqrt{6}$.

The discrete-time logistic model that we have considered in class is from a family of models that can be written as

$$x_{n+1} = f(x_n) = g(x_n)x_n.$$

If x_n is the population size, then $g(x_n)$ can be interpreted as the growth rate (number of offsprings per adult). If $g(x_n) \neq \text{const}$, then the growth rate is dependent on the current population size or *density-dependent*. Different models will assume different type of dependence (form of $g(x)$). Logistic map is one possible model. It has proven useful and is considered classical. However, as we discussed in class, not all of its solutions exhibit logistic growth (that we saw in the data) and for some parameter values the model does not make sense at all. In particular, for $x_n > K$ we have $g(x_n) < 0$, which should not happen. Two models below overcome this by choosing different form of $g(x_n)$.

2. The Beverton-Holt Model. Consider

$$g(x_n) = \frac{r}{1 + \frac{r-1}{K}x_n},$$

with $r > 0$ and $K > 0$.

a) Plot the graph of $g(x_n)$ and notice the difference with the logistic model.

b) Find fixed points of this map.

c) Study analytically the stability of the fixed points. Summarize your results in a bifurcation diagram.

d) Verify your results from b) with cobwebbing (you'll need to draw separate pictures for several regions of r).

e) Use your cobweb diagrams to sketch the solutions

f) Based on your results so far, do you think there can be cycles and chaos in this model?

g) This model is one of the very few nonlinear ones for which the solution can be written explicitly. Use the transformation $u = \frac{1}{x_n}$ to show that

$$x_n = \frac{rx_0}{1 + \frac{r^n - 1}{K}x_0}.$$

3. The Ricker Model. For this model

$$g(x_n) = \exp\left[r\left(1 - \frac{x_n}{K}\right)\right],$$

with $r > 0$ and $K > 0$.

a) Plot the graph of $g(x_n)$ and notice the difference with the logistic model and similarity and difference with the Beverton-Holt model.

b) Find fixed points of this map.

c) Study analytically the stability of the fixed points. Summarize your results in a bifurcation diagram.

d) Do cobwebbing and sketch the solution for some $0 < r < 2$.

e) Look at the setting for the cobwebbing for $r > 2$. Does it resemble the setting for the logistic or Beverton-Holt model? What features determine that? Do you think there will be cycles and chaos in this model as r increases?