Example 7.6.6 Statistics of Spread for the Geometric Distribution with $q = 0.1$

When $q = 0.1$, we have

$$\begin{aligned} \mathrm{Var}(T) &= \frac{1 - 0.1}{0.1^2} = 90 \\ \sigma_T &= \sqrt{90} \approx 9.49 \\ \mathrm{CV} &= \frac{\sigma_T}{\mathrm{E}(T)} \approx 0.949 \end{aligned}$$

The large value of the coefficient of variation indicates that waiting is highly unpredictable, as experience with infrequent phone calls confirms.

We summarize these results in the following theorem.

Theorem 7.16 A random variable T follows the geometric distribution when it measures the time until the first success, where the probability of a success on each step is independent and equal to q. The probability distribution g_t and the cumulative distribution G_t are

$$\begin{aligned} g_t &= q(1-q)^{t-1} \\ G_t &= 1 - (1-q)^t \end{aligned}$$

The mean $\mathrm{E}(T)$, median $\tilde{T}$, and variance σ_T^2 are

$$\begin{aligned} \mathrm{E}(T) &= \frac{1}{q} \\ \tilde{T} &= \frac{\ln(0.5)}{\ln(1-q)} \\ \sigma_T^2 &= \frac{1-q}{q^2} \end{aligned}$$

The geometric distribution can be applied in the same wide array of cases as the binomial distribution because the two result from the same process.

Example 7.6.7 Waiting Time Until an Experiment Exactly Matches a Desired Result

In Example 7.5.6, we studied a case where an experiment exactly matches the most probable result (in this case, of finding exactly 17 out of 50 molecules inside a cell) with a probability of $p \approx 0.1178$. If we repeat this experiment until we achieve that result, the number of replicates T is governed by a geometric distribution with $q \approx 0.1178$. Then

$$\begin{aligned} \mathrm{E}(T) &= \frac{1}{q} \approx 8.487 \\ \tilde{T} &= \frac{\ln(0.5)}{\ln(1-q)} \approx 5.53 \\ \sigma_T^2 &= \frac{1-q}{q^2} \approx 63.54 \\ \sigma_T &\approx 7.971 \end{aligned}$$

The Exponential Distribution

The exponential distribution is the continuous-time version of the geometric distribution. This distribution describes a random variable measuring the *exact* time when a molecule leaves a cell. Finding the distribution requires writing and solving a differential equation.

Differential equations are described by **rates**, and the rate in this problem is the rate at which the molecule leaves. This **probabilistic rate** differs slightly from an ordinary rate. A molecule, after all, is either inside or outside at any given time and does not leak out the way a volume of fluid does. We define a probabilistic rate as a limit in a similar way, however.

Definition 7.10 If an event occurs at a probabilistic rate λ, the probability that it occurs in the short time Δt is approximately $\lambda\Delta t$. More precisely, if $p(\Delta t)$ is the probability that an event occurs in time Δt, then $\lim_{\Delta t\to 0} \frac{p(\Delta t)}{\Delta t} = \lambda$.

Because $p(\Delta t)$ represents a probability, which has no units or dimensions, a probabilistic rate has dimensions of 1/time.

Example 7.6.8 Application of the Definition of a Probabilistic Rate

Suppose an event occurs at a probabilistic rate of $\lambda = 10.0$/s. In the short time $\Delta t = 0.01$ s, the event occurs with probability $10.0 \cdot 0.01 = 0.1$.

Example 7.6.9 The Definition of a Probabilistic Rate Applies Only for Small Values of Δt

If an event occurs at the probabilistic rate of $\lambda = 10.0$/s, and we wish to know the probability that an event occurs during a time $\Delta t = 0.5$ s, the definition gives a probability of $\lambda\Delta t = 10.0 \cdot 0.5 = 5.0$. This is larger than 1 and cannot represent a real probability.

Let $P(t)$ be the probability that the molecule is inside at time t. To write a differential equation, we must compute the probability that it is inside a short time Δt later. Much as in the derivation of the geometric distribution, a molecule is inside at $t + \Delta t$ if time was inside at time t and did not leave in the next Δt. That is,

$$\begin{aligned}(\text{probability inside at } t + \Delta t) &= (\text{probability inside at } t)\\ &\quad\times (\text{probability did not leave in time } \Delta t)\end{aligned}$$

If the molecule leaves at probabilistic rate λ, the definition tells us that

$$\text{probability it did leave in time } \Delta t = \lambda\Delta t$$

for small values of Δt. Therefore,

$$\text{probability it did not leave in time } \Delta t = 1 - \lambda\Delta t$$

For small values of Δt,

$$P(t + \Delta t) = P(t)(1 - \lambda\Delta t)$$

This equation becomes exact as $\Delta t \to 0$. Directly taking the limit of both sides gives

$$\begin{aligned}\lim_{\Delta t\to 0} P(t + \Delta t) &= \lim_{\Delta t\to 0} P(t)(1 - \lambda\Delta t) && \text{take limit of both sides}\\ P(t + 0) &= P(t)(1 - \lambda \cdot 0) && \text{both sides are continuous (Section 2.3), substitute } \Delta t = 0\\ P(t) &= P(t) && \text{evaluate both sides}\end{aligned}$$

This is true, but it tells us nothing about the dynamics of $P(t)$.

Instead, we rearrange the equation by placing all the Δt's on the left-hand side.

$$\begin{aligned}P(t + \Delta t) &= P(t)(1 - \lambda\Delta t) && \text{original equation}\\ P(t + \Delta t) &= P(t) - \lambda\Delta t\, P(t) && \text{multiply out right-hand side}\\ P(t + \Delta t) - P(t) &= -\lambda\Delta t\, P(t) && \text{subtract } P(t) \text{ from both sides}\\ \frac{P(t + \Delta t) - P(t)}{\Delta t} &= -\lambda P(t) && \text{divide both sides by } \Delta t\end{aligned}$$

Upon rearranging, the left-hand side matches the definition of the derivative ("Limits and the Derivative", Section 2.1). Therefore,

$$\lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = \frac{dP}{dt}$$

Thus $P(t)$ obeys the differential equation

$$\frac{dP}{dt} = -\lambda P(t)$$

We have seen this differential equation before, describing the decline of a population (Section 5.4). With the method of separation of variables, we found the solution of this differential equation (Section 5.4) to be

$$P(t) = Ke^{-\lambda t}$$

The constant K must be deduced from the initial condition. In this case, we know for certain that the molecule is inside at $t = 0$, so $P(0) = 1$. Therefore, $P(0) = 1 = Ke^{-\lambda \cdot 0} = K \cdot 1 = K$. The constant $K = 1$, so

$$P(t) = e^{-\lambda t}.$$

We defined $P(t)$ as the probability that the molecule has *not* left by time t. Finding the cumulative distribution function (c.d.f.), and the probability density function (p.d.f.) describing the time T when it leaves, requires a few more steps. If the molecule has not left at some time t, then the time T at which it does leave must be greater than t, or $T > t$. Therefore,

$$P(t) = \Pr(T > t)$$

The c.d.f. $F(t)$, is defined by

$$F(t) = \Pr(T \leq t)$$

the probability that $T \leq t$. Therefore,

$$F(t) = 1 - P(t) = 1 - e^{-\lambda t}$$

The probability density function is the **derivative** of the cumulative distribution function (Section 6.6), so

$$f(t) = F'(t) = \lambda e^{-\lambda t}$$

We summarize these results in a theorem.

Theorem 7.17 Suppose an event happens at a constant rate λ. The random variable T measuring the time until the first event has probability density function $f(t)$ and cumulative distribution function $F(t)$ with formulas

$$f(t) = \lambda e^{-\lambda t}$$
$$F(t) = 1 - e^{-\lambda t}$$

This is called the exponential distribution.

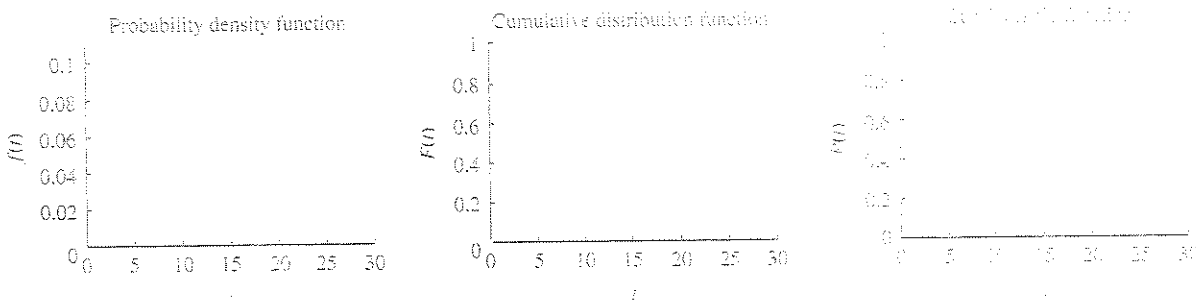
Example 7.6.10 The Behavior of the Exponential Distribution with $\lambda = 0.1$

Suppose that a molecule leaves at rate $\lambda = 0.1$. Then the probability density function is

$$f(t) = 0.1e^{-0.1t}$$

(Figure 7.6.42a). This is the p.d.f. we studied in Examples 6.7.7 and 6.7.11. The cumulative distribution function is

$$F(t) = 1 - e^{-0.1t}$$



The functions describing the exponential distribution with $\lambda = 0.1$

(Figure 7.6.42b), which rises slowly from 0 to 1. Finally, $P(t)$, the probability that [illegible] molecule is still inside, is

$$P(t) = e^{-0.1t}$$

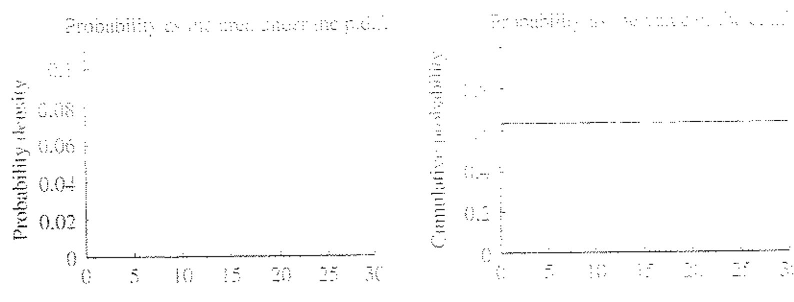
(Figure 7.6.42c), which decreases slowly from 1.

Computing Probabilities Associated with the Exponential Distribution with $\lambda = 0.1$

To find probabilities using the p.d.f., we must integrate. For example, the probabil[illegible] that the molecule leaves between times 0 and 10 is

$$\begin{aligned} \Pr(0 \le T \le 10) &= \int_0^{10} 0.1e^{-0.1t}\,dt \\ &= -e^{-0.1t}\Big|_0^{10} \\ &= -e^{-1.0} + e^{-0.0} \approx 0.632 \end{aligned}$$

(Figure 7.6.43a).



Computing probabilities with the p.d.f. and the c.d.f.

This matches the value we find with the cumulative distribution function, bec[illegible]

$$\begin{aligned} \Pr(0 \le T \le 10) &= \Pr(T \le 10) \\ &= F(10) \\ &= 1 - e^{-1.0} \approx 0.632 \end{aligned}$$

(Figure 7.6.43b).

The Behavior of the Exponential Distribution with $\lambda = 10.0$

At the greater rate $\lambda = 10.0$, the probability density function [illegible] drops off rapidly (Figure 7.6.44a). [illegible] creases to [illegible] (Figure [illegible]

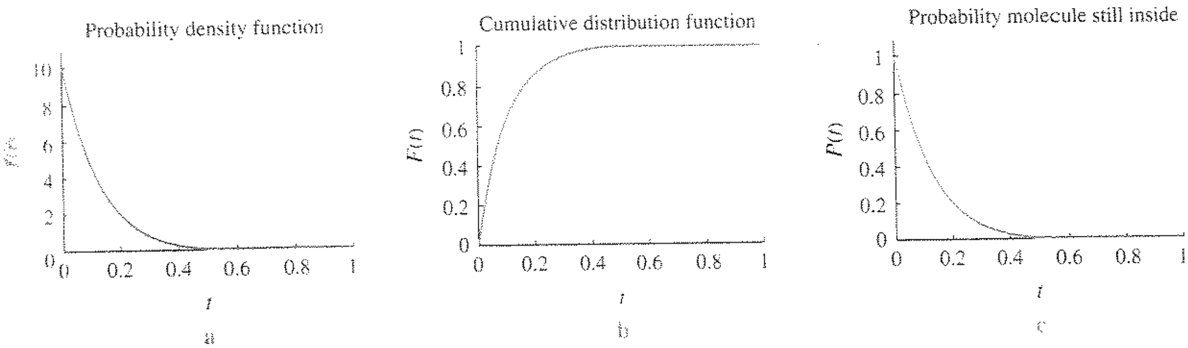


FIGURE 7.6.44

Probability density function, cumulative distribution function, and survivorship function with $\lambda = 10.0$

Using the probability density function, we can find the basic statistics that describe this distribution.

Theorem 7.18 Suppose the random variable T follows an exponential distribution with parameter λ. Then

$$\mathrm{E}(T) = \frac{1}{\lambda}$$

$$\mathrm{Var}(T) = \frac{1}{\lambda^2}$$

$$\hat{T} = \frac{\ln(2)}{\lambda}$$

Proof: The expectation of T is

$$\begin{aligned}
\mathrm{E}(T) &= \int_0^\infty t f(t) dt \\
&= \int_0^\infty t\lambda e^{-\lambda t} dt \\
&= -\frac{\lambda t e^{-\lambda t} + e^{-\lambda t}}{\lambda}\Bigg|_0^\infty \\
&= \lim_{st\to\infty} -\frac{\lambda s e^{-\lambda s} + e^{-\lambda s}}{\lambda} + \frac{-1}{\lambda} \\
&= \frac{1}{\lambda}
\end{aligned}$$

where the integral was evaluated using integration by parts, checked in Exercise 33. Integration by parts can again be used to compute the variance of T (Exercise 34) as follows:

$$\begin{aligned}
\mathrm{Var}(T) &= \int_0^\infty t^2 f(t) dt - \mathrm{E}(T)^2 \\
&= -\frac{\lambda^2 t^2 e^{-\lambda t} + 2\lambda t e^{-\lambda t} + 2e^{-\lambda t}}{\lambda^2}\Bigg|_0^\infty - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$