From: A course in Mathematical Biology (SIAM books) by de Vries et al.

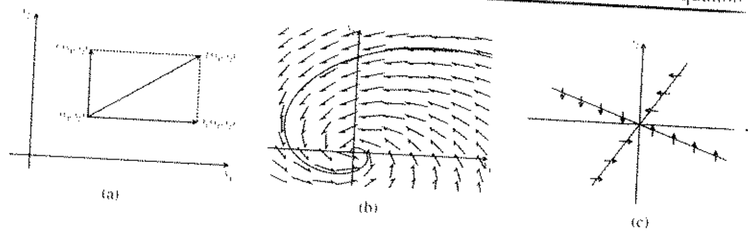

Figure 3.6. (a) *Construction of the vector field for (3.14); (b) the direction field for Example 3.4.2 and one typical trajectory; (c) the nullclines with vector field at the nullclines.*

3.4 Qualitative Analysis of 2 × 2 Systems

In this section, we develop a qualitative theory for systems of two differential equations, much in the spirit of Section 3.2, where we introduced phase-line and vector-field analysis. Here, we will use *phase-plane* analysis, *vector-field* analysis, and the *phase portrait*. With these methods, the qualitative behavior of a system of equations can be understood without solving the equations explicitly. Explicit solution methods can be found in other textbooks on ODEs (such as Boyce and DiPrima [25]).

Consider a system of two differential equations,

$$x_1' = f_1(x_1, x_2),$$
$$x_2' = f_2(x_1, x_2). \tag{3.14}$$

At each $x = (x_1, x_2) \in \mathbb{R}^2$, the *vector field* $f(x) = (f_1(x), f_2(x))$ represents a vector, as shown in Figure 3.6. A solution $x(t) = (x_1(t), x_2(t))$ represents a parametric curve in the (x_1, x_2) plane, called a *trajectory* or an *orbit*, whose tangent vector $x'(t) = (x_1'(t), x_2'(t))$ is specified by the *vector field* $f(x(t)) = (f_1(x_1(t), x_2(t)), f_2(x_1(t), x_2(t)))$. We can obtain a good impression of the overall dynamics if we plot many vectors in the (x_1, x_2) plane. For each chosen point (x_1, x_2), we calculate $(f_1(x_1, x_2), f_2(x_1, x_2))$ and sketch this vector. In Figure 3.6 (a), we show how to calculate one such vector. We repeat this procedure at many different points until the whole plane is filled with vectors. It is sometimes convenient to consider only the direction of vector field and not the magnitude. This yields a *direction field* for the system (Figure 3.6 (b)).

Since solution curves are tangential to the vector field, f, we often can follow trajectories just by following the arrows. In Figure 3.6 (b), a typical solution curve is shown (in this case, we have a spiral converging to the stable origin). The vector field can be used to sketch more than one typical solution, starting at different initial conditions. The sketch of the (x_1, x_2) plane with a number of typical solutions is called a *phase portrait*. Of course, "typical" is a rather vague notion and you need some experience to be able to decide which solutions represent the qualitative behavior. We will demonstrate and practice this in what follows.

Many computer packages provide a routine to draw the vector field and the phase portrait of an ODE system. In Chapter 8, we will learn how to do this with Maple.

Another helpful tool for obtaining insight into the phase portrait are *nullclines* (or *isoclines*). The x_1-nullcline, n_1, is the set of points (x_1, x_2) such that $x_1' = f(x_1, x_2) = 0$, that is,

$$n_1 := \{(x_1, x_2) | f_1(x_1, x_2) = 0\}.$$

Similarly, the x_2-nullcline, n_2, is

$$n_2 := \{(x_1, x_2) | f_2(x_1, x_2) = 0\}.$$

On the x_1-nullcline, n_1, all vectors of the vector field are vertical (since $x_1' = 0$). Similarly, on n_2, all vectors are horizontal (since $x_2' = 0$). At intersections of n_1 and n_2, we have $x_1' = 0$ and $x_2' = 0$. Hence a *steady-state* or *equilibrium point* exists at any intersection of n_1 and n_2. In Figure 3.6 (c), we show the nullclines corresponding to the vector field of Figure 3.6 (b).

In general, *equilibria*, or *steady states* of (3.14) are solutions of

$$f_1(x_1, x_2) = 0, \qquad f_2(x_1, x_2) = 0,$$

which we denote by $(\bar{x}_1, \bar{x}_2)$. The steady states play an important role in the understanding of the whole dynamics. In many cases, if the behavior near each steady state is known, then the global behavior of solutions can be understood quite well. It turns out that we can classify all possible behaviors which can occur near a steady state. We will do so in the following two sections. In Section 3.4.1, we first treat specific linear systems. After that, we generalize to arbitrary linear systems. In Section 3.4.2, we consider nonlinear systems. Phase-plane analysis will then be applied to the population interaction model (in Section 3.4.3) and to the epidemic model (in Section 3.4.4).

3.4.1 Phase-Plane Analysis: Linear Systems

Step 1: Specific Linear Systems

(1a) Real Eigenvalues

Consider the simplest linear system,

$$x_1' = \lambda_1 x_1,$$
$$x_2' = \lambda_2 x_2, \tag{3.15}$$

whose unique steady state is the origin, $(\bar{x}_1, \bar{x}_2) = (0, 0)$. In matrix form, we can write

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Note that λ_1 and λ_2 are the *eigenvalues* of the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Solutions to (3.15) are

$$x_1(t) = x_1(0)e^{\lambda_1 t}, \qquad x_2(t) = x_2(0)e^{\lambda_2 t}.$$

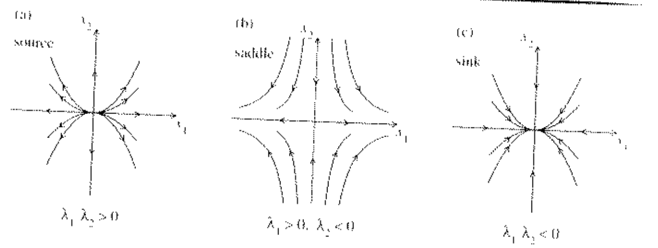

Figure 3.7. *Three qualitatively different phase portraits for system (3.15) depending on the sign pattern of λ_1 and λ_2. (a) $\lambda_1, \lambda_2 > 0$; (b) $\lambda_1 > 0$, $\lambda_2 < 0$; (c) λ_1, λ_2 [illegible]. Here it is assumed that λ_2 is the larger eigenvalue when sketching (a) and (c).*

Plotting the parametric curves $(x_1(t), x_2(t))$ for different initial values $(x_1(0), x_2(0))$, w[e] arrive at three distinct phase portraits, depending on the signs of λ_1 and λ_2, as shown in Figure 3.7.

Case (a): If both eigenvalues λ_1 and λ_2 are positive, then all solutions diverge from the steady state (0, 0). In Figure 3.7 (a), several trajectories are shown for positive, negative, or mixed initial conditions. In this case, the steady state (0, 0) is called a *source* or an *unstable node*.

Case (b): If the eigenvalues have opposite signs, $\lambda_1 > 0$ and $\lambda_2 < 0$, say, then $x_1(t)$ is exponentially increasing, while $x_2(t)$ is decreasing. All solutions approach the x_1-axis, as shown in Figure 3.7 (b). In this case, the steady state (0, 0) is called a *saddle*.

Case (c): If both eigenvalues are negative, then all solutions converge to the steady state (0, 0), as shown in Figure 3.7 (c). The steady state is called a *sink* or *stable node*.

(1b) Complex Eigenvalues

Consider the linear system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3.16}$$

For $\beta \neq 0$, the system has the origin, (0, 0), as its only steady state. The coefficient matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right)$ has two complex conjugate eigenvalues

$$\lambda_1 = \alpha + \beta i \quad \text{and} \quad \lambda_2 = \alpha - \beta i.$$

We can verify (see the exercises) that (3.16) has two special solutions, namely,

$$x^{(1)}(t) = e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \qquad x^{(2)}(t) = e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

The superposition principle of linear systems implies that all solutions to (3.16) are of the form

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) = a e^{\alpha t}\begin{pmatrix} \cos(\beta t + \phi) \\ -\sin(\beta t + \phi) \end{pmatrix}$$

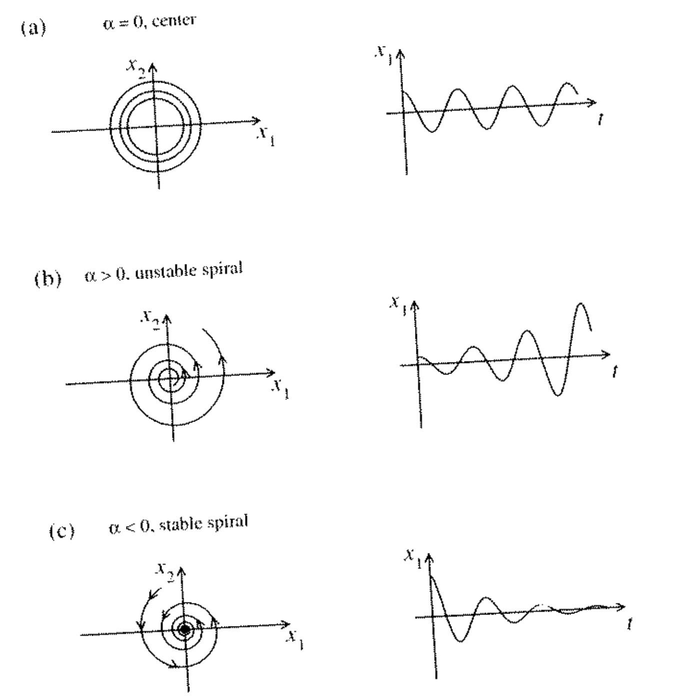

Figure 3.8. *Three qualitatively different cases for system (3.16), depending on the value of the parameter α. (a) $\alpha = 0$; (b) $\alpha > 0$; (c) $\alpha < 0$. Graphs in the left column show phase portraits. Graphs in the right column show a typical solution for $x_1(t)$. Here it is assumed that $\beta > 0$ when sketching (a)–(c), so that the spirals move clockwise.*

or

$$\begin{aligned} x_1(t) &= a e^{\alpha t}\cos(\beta t + \phi), \\ x_2(t) &= -a e^{\alpha t}\sin(\beta t + \phi), \end{aligned} \tag{3.17}$$

where a and ϕ are determined by the initial conditions, $(x_1(0), x_2(0))$.

Using (3.17), we can classify three distinct cases.

Case (a): $\alpha = 0$, so that both eigenvalues are purely imaginary. All solutions are periodic, and all trajectories are closed orbits surrounding the steady state (0, 0), as shown in Figure 3.8 (a). The steady state is called a *center*.

Case (b): $\alpha > 0$, so that both eigenvalues have positive real parts. The exponential function $e^{\alpha t}$ grows for $t > 0$. All trajectories spiral away from the steady state (0, 0), as shown in Figure 3.8 (b). The steady state is called an *unstable spiral* or a *spiral source*.

Case (c): $\alpha < 0$, so that both eigenvalues have negative real parts. The exponential function $e^{\alpha t}$ decays for $t > 0$. All trajectories spiral towards the steady state $(0, 0)$, as shown in Figure 3.8 (c). The steady state is called a *stable spiral* or a *spiral sink*.

Corresponding solutions $x_1(t)$ for each case are shown in Figure 3.8.

Step 2: General Linear Systems

We now consider a general linear system,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.18}$$

If we make the transformation of coordinates

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where P is a 2×2 invertible matrix, then $y = (y_1, y_2)$ satisfies the system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = B\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{3.19}$$

where $B = P^{-1}AP$. The matrix B is *similar* to A—it has identical eigenvalues (Hirsch and Smale [86]). Hence,

Systems (3.18) and (3.19) have the same phase portraits.

It is known from linear algebra (see [106]) that if A has two distinct real eigenvalues λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$, then we can choose P such that

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

If A has two complex conjugate eigenvalues $\lambda_1 = \bar{\lambda}_2 = \alpha + \beta i$, then we can choose P such that

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Thus, we conclude that the phase portraits of (3.18) will be the same as those of systems (3.15) or (3.16), studied earlier. Before presenting a theorem about the stability of the origin, we work out the details of computing the matrix B for two specific examples.

Example 3.4.1: Consider the linear system

$$\dot{x} = 2x - 2y,$$
$$\dot{y} = 2x - 3y. \tag{3.20}$$

In vector matrix notation, we have

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

It is straightforward to verify that the eigenvalues and corresponding eigenvectors of A are $\lambda_1 = 1$, $\zeta_1 = \binom{2}{1}$ and $\lambda_2 = -2$, $\zeta_2 = \binom{1}{2}$.

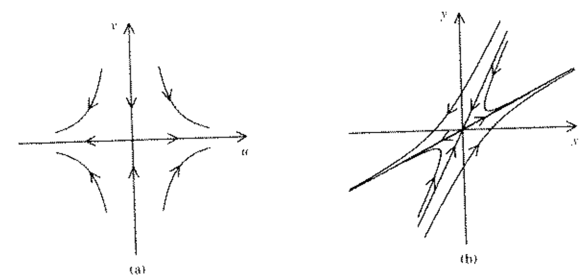

Figure 3.9. *The phase portraits of* (a) (3.21) *and* (b) (3.20).

The eigenvalues of A are real and distinct. If we use the eigenvectors ζ_1 and ζ_2 as columns of a matrix P, we obtain the transformation

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

then

$$B = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

From the solution of the related linear system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.21}$$

we can recover the solution of (3.20) via

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix}.$$

The phase portrait of (3.21) is shown in Figure 3.9 (a), and the corresponding phase portrait of (3.20) is shown in Figure 3.9 (b). The transformation P maps the unstable direction $\binom{1}{0}$ of (3.21) onto the unstable direction $\binom{2}{1}$ of (3.20). Similarly, the stable direction $\binom{0}{1}$ of (3.21) is mapped onto the stable direction $\binom{1}{2}$ of (3.20). Note that the phase portrait shown in Figure 3.9 (b) is a compressed and rotated version of the phase portrait shown in Figure 3.9 (a).

Example 3.4.2: We consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of the corresponding matrix are $\lambda_1 = -1 + 2i$ and $\lambda_2 = \bar{\lambda}_1 = -1 - 2i$. The corresponding (complex) eigenvectors are

$$\zeta_1 = \begin{pmatrix} -4 \\ -2 + 2i \end{pmatrix} \quad \text{and} \quad \zeta_2 = \bar{\zeta}_1.$$

We write $\zeta_1 = \phi + i\psi$ with real vectors ϕ and ψ and obtain the transformation matrix $P = (\phi\psi)$ (see Perko [132]), namely,

$$P = \begin{pmatrix} -4 & 0 \\ -2 & 2 \end{pmatrix},$$

with inverse

$$P^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix}.$$

Using this transformation P, we obtain the matrix

$$B = P^{-1}AP = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix},$$

and the transformed system has the form of (3.16):

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The solution can be written in the general form

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{-t}\left(c_1\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}\right).$$

The solution describes oscillations around (0, 0) with frequency π^{-1}, where the amplitude decays exponentially like e^{-t}. Hence solutions converge to (0, 0) and the steady state (0, 0) is a stable spiral. The vector field and one solution curve were shown in Figure 3.6 (b).

In all the cases discussed above, solutions only converge to the steady state at (0, 0) when both eigenvalues $\lambda_1, \lambda_2 < 0$ (the origin is a stable node), or when the real part of the eigenvalues satisfies $\alpha < 0$ (the origin is a stable spiral). When solutions converge to the steady state, we say the steady state is *asymptotically stable*.

We have seen that we can classify the equilibria of a linear system according to the eigenvalues of the corresponding coefficient matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Sometimes it is more convenient to use two other characteristic values of A, namely, the *trace*, $\operatorname{tr} A = a + d$, and the *determinant*, $\det A = ad - bc$. It is known that the trace is always the sum of the eigenvalues, $\operatorname{tr} A = \lambda_1 + \lambda_2$, and the determinant is the product, $\det A = \lambda_1\lambda_2$. Moreover, one can use the trace and determinant to calculate the eigenvalues. In the exercises, the reader is asked to show that

$$\lambda_{1,2} = \frac{\operatorname{tr} A}{2} \pm \frac{1}{2}\sqrt{(\operatorname{tr} A)^2 - 4\det A}. \tag{3.22}$$

Note that the formula in (3.22) holds only for 2×2 matrices. For higher-order matrices, there is no simple formula of this form.

From (3.22), we see that it is necessary to have $\operatorname{tr} A < 0$ in order to have a steady state that is asymptotically stable (otherwise at least one eigenvalue would have a positive real part). If $\operatorname{tr} A < 0$, then the discriminant, $(\operatorname{tr} A)^2 - 4\det A$, is either negative or smaller than $(\operatorname{tr} A)^2$. Hence the real part of the eigenvalues is always negative, and (0, 0) is asymptotically stable. We can summarize our conclusions in the following theorem.

Theorem 3.3. *For a linear system,* (3.18), *the following are equivalent:*

- *the equilibrium* (0, 0) *is asymptotically stable;*
- *all eigenvalues of A have negative real parts;*
- $\det A = ad - bc > 0$ *and* $\operatorname{tr} A = a + d < 0$.

We can treat all different combinations for the sign of trace and determinant and obtain a complete picture of possible behavior near an equilibrium point. Figure 3.10 shows the "zoo" of all possible types of behavior for steady states of two-dimensional systems.

We can summarize the possible types of behavior as follows:

1. Case $\det A < 0$. Then $(\operatorname{tr} A)^2 - 4\det A > (\operatorname{tr} A)^2$. From formula (3.22), it follows that there is one positive and one negative eigenvalue, $\lambda_1 > 0$ and $\lambda_2 < 0$, say. Hence, (0,0) is a saddle point. Moreover, solutions grow as $e^{\lambda_1 t}$ in the direction of the eigenvector φ_1 corresponding to λ_1, and solutions decay as $e^{\lambda_2 t}$ in the direction of the eigenvector φ_2 corresponding to λ_2. In Figure 3.10, the *stable* and *unstable* eigenvectors are shown.

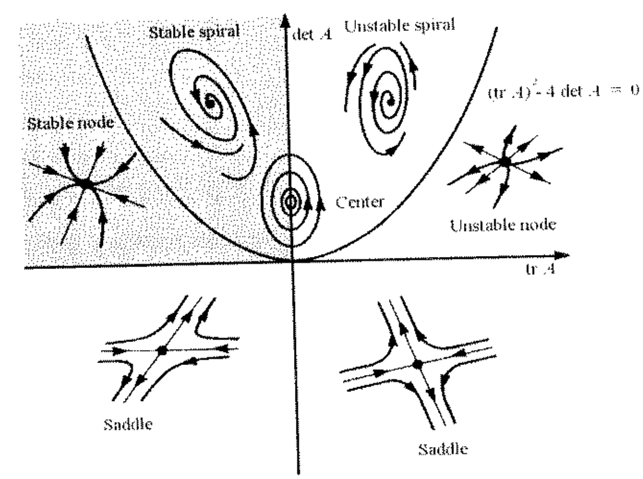

Figure 3.10. *The zoo for the general linear system,* (3.18). *This is a modified version of Figure 5.14 in Edelstein-Keshet* [51].

2. Case $\det A > 0$, $\operatorname{tr} A < 0$. If $(\operatorname{tr} A)^2 < 4 \det A$ (above the parabola in Figure 3.10), then λ_1, λ_2 are complex conjugate eigenvalues with real part $\frac{\operatorname{tr} A}{2} < 0$, and $(0,0)$ is a stable spiral. If $(\operatorname{tr} A)^2 > 4 \det A$ (below the parabola), then λ_1, λ_2 are real, but they have the same sign, and $(0,0)$ is a stable node.

3. Case $\det A > 0$, $\operatorname{tr} A > 0$. Depending on the sign of $(\operatorname{tr} A)^2 - 4 \det A$, we have either an unstable spiral or an unstable node.

4. Case $\det A > 0$, $\operatorname{tr} A = 0$. In this case we have a center.

5. The remaining cases ($\det A = 0$ or $(\operatorname{tr} A)^2 - 4 \det A = 0$) will not be discussed. We refer to Hirsch and Smale [86] for these cases.

3.4.2 Nonlinear Systems and Linearization

Consider a nonlinear system in $\mathbb{R}^2$,

$$x_1' = f_1(x_1, x_2),$$
$$x_2' = f_2(x_1, x_2), \tag{3.23}$$

where f_1 and f_2 are continuously differentiable functions.

In general, each pair $(\bar{x}_1, \bar{x}_2)$ satisfying $f_1(\bar{x}_1, \bar{x}_2) = f_2(\bar{x}_1, \bar{x}_2) = 0$ is called an equilibrium or a steady state for (3.23). We would like to understand the behavior of the solutions near equilibria.

For linear systems, we observed that solutions converge to $(0,0)$, they diverge away from $(0,0)$, or, in the center case, they stay close by. Before we can generalize these observations to nonlinear systems, we need some definitions from dynamical systems theory (see Perko [132]).

Definition 3.4.

(a) *A steady state $(\bar{x}_1, \bar{x}_2)$ is called* stable *if a solution which starts nearby stays nearby.*

More formally: $(\bar{x}_1, \bar{x}_2)$ is stable if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that solutions to initial data (x_1^0, x_2^0) with $\|(x_1^0, x_2^0) - (\bar{x}_1, \bar{x}_2)\| < \delta$ satisfy $\|(x_1(t), x_2(t)) - (\bar{x}_1, \bar{x}_2)\| < \varepsilon$ for all time $t > 0$. Here, $\|.\|$ denotes the Euclidean vector norm.

(b) *A steady state $(\bar{x}_1, \bar{x}_2)$ which is not stable is called* unstable *(there is at least one solution which diverges from $(\bar{x}_1, \bar{x}_2)$).*

(c) *A steady state $(\bar{x}_1, \bar{x}_2)$ is called* asymptotically stable *if $(\bar{x}_1, \bar{x}_2)$ is stable and all solutions near $(\bar{x}_1, \bar{x}_2)$ converge to $(\bar{x}_1, \bar{x}_2)$.*

More formally: $(\bar{x}_1, \bar{x}_2)$ is asymptotically stable if $(\bar{x}_1, \bar{x}_2)$ is stable, and there exists a $\delta > 0$ such that all solutions with initial data (x_1^0, x_2^0), with $\|(x_1^0, x_2^0) - (\bar{x}_1, \bar{x}_2)\| < \delta$, satisfy $\lim_{t\to\infty} \|(x_1(t), x_2(t)) - (\bar{x}_1, \bar{x}_2)\| = 0$.

We can determine the stability of a steady state $(\bar{x}_1, \bar{x}_2)$ by linearizing (3.23). The process is similar to the linearization of discrete-time systems, treated in Section 2.3.2.

Let

$$x_1(t) = \bar{x}_1 + z_1(t),$$
$$x_2(t) = \bar{x}_2 + z_2(t),$$

where $z_1(t)$ and $z_2(t)$ are assumed to be small, so that they can be thought of as perturbations to the steady state. We denote $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and $z = (z_1, z_2)$, and write the Taylor expansion of $f = (f_1, f_2)$ about $(\bar{x}_1, \bar{x}_2)$:

$$f(\bar{x} + z) = f(\bar{x}) + Df(\bar{x}) \cdot z + \text{higher-order terms},$$

where

$$Df(\bar{x}_1, \bar{x}_2) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \Bigg|_{(x_1, x_2) = (\bar{x}_1, \bar{x}_2)}$$

contains the partial derivatives of f evaluated at $(\bar{x}_1, \bar{x}_2)$ (for a reminder on partial derivatives, see Section 4.1). The matrix $Df(\bar{x}_1, \bar{x}_2)$ is called the *Jacobian matrix* of f at $(\bar{x}_1, \bar{x}_2)$.

We substitute the Taylor expansion into (3.23) and we drop the higher-order terms. Since $x_1' = \frac{d}{dt}(\bar{x}_1 + z_1(t)) = z_1'$ and $x_2' = z_2'$, and since $f(\bar{x}) = 0$, we obtain a linear system governing the dynamics of the perturbation (z_1, z_2):

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.24}$$

We know already from the previous section how to treat linear systems. For most (but not all) steady states, conclusions obtained for the linearized system indeed carry over to the original nonlinear system.

Definition 3.5. *$(\bar{x}_1, \bar{x}_2)$ is called* hyperbolic *if all eigenvalues of the Jacobian $Df(\bar{x}_1, \bar{x}_2)$ have nonzero real part.*

Theorem 3.6 (Hartman–Grobman). *Assume that $(\bar{x}_1, \bar{x}_2)$ is a hyperbolic equilibrium. Then, in a small neighborhood of $(\bar{x}_1, \bar{x}_2)$, the phase portrait of the nonlinear system,* (3.23), *is equivalent to that of the linearized system,* (3.24).

Remark 3.4.1.

1. By Theorems 3.3 and 3.6, at a hyperbolic equilibrium $\bar{x}$, stability properties are determined by the eigenvalues of the Jacobian matrix, $Df(\bar{x}_1, \bar{x}_2)$. This method of linearization may fail for nonhyperbolic equilibria.

2. The phrase "equivalent to" in the above theorem refers to *topological equivalence* of vector fields. This means that in a neighborhood of $(\bar{x}_1, \bar{x}_2)$, there is a homeomorphism (a continuous one-to-one map between open sets) which maps the vector field of the nonlinear system to the vector field of its linearization. In that case, the phase portrait near the stationary point is one of those shown in Figure 3.10. The theory behind the Hartman–Grobman theorem is given in Perko [132].

   For an example, recall Example 3.4.1. The two phase portraits in Figure 3.9 are topologically equivalent, and the homeomorphism is given by the matrix P.