Worksheet 8 - Applications of Integration
Due Wednesday November 4th, 2015

0. This worksheet is going to feel a bit harder than the previous ones! There are more "proof"-type problems, so why not work on this worksheet with a group!

1. **Surface Area Definition:** If you read Section 13.6 of the book (which you should!) you can learn all about surface area. A lot of the definitions and derivations are things that will come up naturally in the context of surface integrals which we'll do in November. So I'll delay the derivations, and just declare that if you have a region S in the xy plane and on top of it a "tent" determined by a function $z = f(x, y)$ then the surface area of that tent is

$$SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

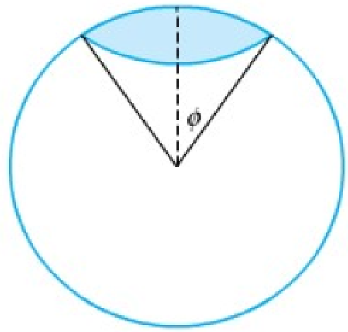
Remember, this is surface area, not volume.

A. (6 points) As a reality check, suppose that we wanted to find the volume and surface area of a hemisphere of radius R. Write down integrals that represent the surface area and volume. Both integrals should be over the same region S in the xy-plane, but the functions you integrate will be different.

B. (6 points) Using polar coordinates evaluate both of these integrals. You should obtain familiar formulas

$$V = \frac{2}{3}\pi R^3,\ SA = 2\pi R^2.$$

C. (6 points) Suppose instead that you didn't want all of the hemisphere. Instead, you wanted just a small sliver around the north pole. Say, the polar cap determined by an angle ϕ. (See below) Show that this region has area $2\pi R^2(1-\cos\phi)$. (Hint: This is nearly identical to the integral in part B. In fact, to solve this problem, you'll just have to change one of the bounds in your iterated integral. This can save you a lot of work. Think about the shadow!)



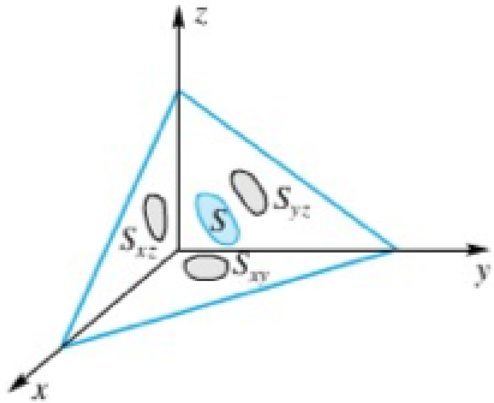
2. **GOATS!** (15 Points) Goats are a very important part of life. I had some goats when I was a kid (named Brady and Bo) and if there's anything that's true about goats, it's that they love to graze on grass (and pine needles). In the following problem you'll compute exactly how much grass a goat can eat assuming it's tethered to a point in the plane or on a sphere! Since our world is roughly a sphere, this is clearly useful.

The Problem Four goats have grazing areas A, B, C, D respectively. The first three goats are each tethered by ropes of length b, the first on a flat plane. The second on the **outside** of a sphere of radius a and the third on the **inside** of a sphere of radius a. The fourth goat rejects the notion of being tied down, and has instead opted to live inside a ring of radius b that has been dropped over a sphere of radius a. (Like a crown). Assume that $b < a$. Determine the formulas for A, B, C, D and arrange them in order of size.

3. **Shadows - In Honor of Hallowe'en!**[1] (10 Points) Assume that the region S in the figure below lies in the plane $z = ax + by + c$ and that S is above the xy-plane. In other words, assume that S_{xy} is the shadow underneath S. Then show that the volume of the solid cylinder under S is

$$A(S_{xy})f(\overline{x}, \overline{y})$$

where $(\overline{x}, \overline{y})$ is the centroid (center of mass) of S_{xy}.



[1]This is a reference to the Agatha Christie Novel *Hallowe'en Party* which is not one of the better Hercule Poirot mysteries

4. **Pappus' Theorem Part 2** Recall that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^2$ then the line between them is the set of all points of the form $\mathbf{u} + t \cdot (\mathbf{v} - \mathbf{u})$.

A. (3 Points) For what values of t will the corresponding point on this line lie between $\mathbf{u}$ and $\mathbf{v}$?

B. (3 points) Notice that we could also write this line as $(1-t)\mathbf{u} + t\mathbf{v}$. Using your answer to part A deduce that every point on the line between $\mathbf{u}$ and $\mathbf{v}$ is of the form

$$a\mathbf{u} + b\mathbf{v}$$

where $a, b \geq 0$ and $a + b = 1$. Combinations like this are called "convex combinations of $\mathbf{u}$ and $\mathbf{v}$." (Hint: The only difficult part of this problem is wrapping your brain around the conversion from t and $1 - t$ to a and b. Give it some thought! If you're stuck, you might just be over-thinking it.)

C. (9 points) Suppose that A and B are two disjoint regions (say puddles of oil) in $\mathbb{R}^2$. Denote by $m(A)$ and $m(B)$ their masses, and by $\mathbf{C}_A$ and $\mathbf{C}_B$ the vectors of their centers of mass. In other words, $\mathbf{C}_A = (\overline{x}, \overline{y})$ where $\overline{x}$ and $\overline{y}$ are the coordinates of the center of mass of A. Prove that the center of mass of the region S consisting of the union of the regions, $S = A \cup B$ has center of mass on the line joining $\mathbf{C}_A$ and $\mathbf{C}_B$. To do this, you will need to find constants $a, b \geq 0$ such that $a + b = 1$ with

$$\mathbf{C}_{A \cup B} = a\mathbf{C}_A + b\mathbf{C}_B.$$

(Hint: This problem has a lot of notation, and one of the challenges is to decipher everything. I'd recommend writing everything out very carefully in terms of integrals, moments, etc. Then try to figure out what a and b are.)