Worksheet 2 - Planes and Cross Products
Due Wednesday Sept. 9, 2015

You are greatly encouraged to work on this worksheet in groups - in fact you can turn in one worksheet per group, provided it is written neatly.

Today, we will wrap up and review our work with vectors in $\mathbb{R}^n$.

0: (Don't turn this problem in) It's important to make sure everyone is on the same page with the basics, so spend a few minutes on these easier problems and make sure everyone in your group gets the same answer.

a) Compute the projection of $< 5, 1 >$ onto the vector $< -10, 3 >$.

b) Suppose that u and v have magnitude 5 and 7 respectively. Suppose that $\mathbf{u} \cdot \mathbf{v} = 12$, what is $\|\text{pr}_{\mathbf{v}}\mathbf{u}\|$?

c) Compute the determinant of

$$\begin{pmatrix} 0 & 1 & -2 \\ 3 & 2 & 2 \\ -1 & 1 & 4 \end{pmatrix}$$

d) Compute the cross product of the vectors $< 0, 1, 2 >$ and $< 2, 1, 0 >$.

As a group, take a look at the following three problems and decide which one you want to work on first. They are all of a different flavor, but all about the same difficulty.

1: With your group, you will work out the formula for the distance from a point $P = (x_0, y_0, z_0)$ to the plane defined by the equation $Ax + By + Cz = D$. The goal of this exercise is to first model this question so that the answer can be stated in terms of projections, and then computing what the formula should be.

(a) First, draw a picture of this situation. I'd recommend drawing the plane, with a normal vector $\mathbf{n}$ and then also the point P which is not on the plane.

(b) Now, write down a formula using our projection notation for the length of the point to the plane, $(\text{pr}_{\mathbf{v}}\mathbf{u})$

(c) Finally, work out your answer to obtain something in terms of $A, B, C, D, x_0, y_0, z_0$.

2: Recall that in class we proved that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$.

a) Use this to write a proof that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. (This is called the Cauchy-Schwarz Inequality) (Hint: don't overthink it, it's really just a sentence.)

b) Now prove the **Triangle Inequality**:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|,$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. (HINT: Compute using the fact that $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$. Now try to use the inequality from part a).)

c) Draw a picture to explain why this is called the Triangle Inequality.

3: In this section will will emphasize some important properties of the cross-product.

a) The most important property of the cross product is that $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Write down two equations (one for $\mathbf{u}$ and one for $\mathbf{v}$) that express this fact.

b) In the book, Theorem C (page 577) contains 6 different properties about cross products. The book says "Proving this theorem is a matter of writing everything out in terms of components..." Verify number 4. on the list.

c) The number $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the triple scalar product of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. Its absolute value

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

is equal to the volume of the parallelepiped determined by the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$. Explain why this fact proves that

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|.$$

d) Let $\mathbf{u}$ and $\mathbf{v}$ be nonparallel vectors and let c be any nonzero vector (all in $\mathbb{R}^3$). Show that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is a vector in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. (Hint: Remember that being in a plane means that it's perpendicular to the normal vector...)

Extra Credit: For an extra 6 homework points, try # 37 and #38 in Section 11.5. (If you have a different edition of the book - send me an email for the correct problems).