RECONSTRUCTING A GRAPH FROM A COLLECTION OF SUBGRAPHS

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F. Harary[1] delivered a lecture in 1963 at a Symposium on graph theory held in Smolenice. In his lecture and subsequent notes, he presents the very interesting idea of reconstructing a graph from a collection of subgraphs. He lets G be a graph with p vertices labeled $\{v_1, \dots, v_p\}$. He defines the subgraph G_i as $G - v_i$ for each $i = 1, \dots, p$. Harary's problem is to reconstruct G using the set $\{G_i | i = 1, \dots, p\}$.

This problem is an extension of one solved by P. J. Kelly[2]. Kelly was able to show that two trees are isomorphic if there are isomorphisms between their respective sets of subgraphs. Kelly also conjected that this is true for any two graphs. F. Harary, hoping that Kelly's conjecture is valid, wishes to actually reconstruct a graph G from its collection of subgraphs. If Kelly's conjecture holds, then indeed Harary may attempt to do so without fear of problems in well-definedness.

Harary's lecture is, for the most part, a rough algorithm for reconstructing the graph G. The algorithm is complete for disconnected graphs, but is incomplete for connected graphs. Harary's plan is as follows:

1. Harary Plan for Reconstructing a Graph from its Subgraphs

1.1. **Find the number of points and lines.** The number of vertices p is the number of cards, or equivalently, one more than the number of vertices in any given card.

The number of edges q is given by

$$q = \frac{\sum_{i=1}^{p} q_i}{p-2}.$$

We must consider only graphs for which $p \geq 3$.

1.2. **Decide whether the graph is connected.** Use Theorem 1 for this purpose.

1.2.1. *If the graph is not connected.* Use a generalization of Statement 2 to find all the components of the graph.

First find the largest components. Then compare a subgraph G_i which contains all such components to each of the subgraphs G_j which does not contain all such components. This yields G directly.

1.2.2. *If the graph is connected.* We do not know how to completely reconstruct a connected graph G. However, we can get the number of cut points r which is the number of subgraphs G_i which are non connected. Also, by Kirchoff, the number of independent cycles of G is $m = q - p + 1$. Then G is a tree iff $m = 0$. If $m > 0$, Harary gives the number of blocks b of G from the number of components k_i of each of the subgraphs G_i.

$$b = 1 + \sum_{i=1}^{p} (k_i - 1)\,.$$

Then Harary gives the connectivity $\pi \geq 1$ of G by

$$\pi = 1 + \min_i \{\pi_i\}.$$

Small Mistakes in Harary

Kelly and Harary define a graph G to be **complement-connected** if both G and $\overline{G}$ are connected.

Harary's Statement 3. A graph G is complement-connected if and only if at least two of the subgraphs G_i are complement-connected.

Harary has overlooked the small counter example P_4 to this statement. P_4 is complement-connected, but none of the subgraphs of P_4 are complement-connected. Perhaps this statement holds for graphs with more than four vertices.

Harary lists in his Problems section a problem which a former student of his claimed to solve in the affirmative. Harary then states that to his knowledge, the claim has not been verified. Indeed the claim is false.

Problem IV. Consider the subset of cards containing graphs G_i obtained from the full set of cards by discarding all but one graph of each isomorphism type. Does this serve to determine the original graph G?

Class of counterexamples for Problem IV. Consider the graphs $H_1 = K_{1,2}$ and $H_2 = K_2 \cup K_1$. Let L be any other graph. Then, if two graphs G_1 and G_2 are such that $G_1 = L \cup H_1$ and $G_2 = L \cup H_2$, then G_1 and G_2 have the same set of cards but are not isomorphic.

Further Analysis

Despite the few small technical problems in Harary's lecture, the ideas are very interesting. We hope to extend these ideas even further. Harary and Kelly hope to take a set of subgraph cards on which are drawn these subgraphs of G and use them to reconstruct G. We would like to give Harary and Kelly any set of cards with graphs and ask whether or not they are the subgraph cards of some graph G. Let's provide some formalism here.

A **card** is a graph. A **card of** G is a subgraph of G obtained by removing a single vertex from G. A T**-card** is a card which is a tree.

We make the definition of card so as to help distinguish that which we are given from that which we wish to construct. We are given cards, we wish to construct a graph.

A **deck** is a set of cards all with the same number of vertices. A deck which is exactly the set of cards of G is called the **deck of** G. If all but one of each isomorphism type is discarded from the deck of G, what remains is the **proper deck of** G. In general, a **proper deck** is a deck in which there is at most one of each isomorphism type. A T**-deck** is a deck of T-cards.

A deck of cards $\mathcal{G}$ is a **constructing deck** if there is a graph G such that $\mathcal{G}$ is the deck of G. We may also say that $\mathcal{G}$ **constructs** G, or equivalently G **deconstructs to** $\mathcal{G}$ If $\mathcal{G}$ is a proper deck and constructs, then $\mathcal{G}$ is a **proper constructing deck**. A T-deck which constructs a tree is called a T**-constructing** T**-deck**. Note that if a deck is T-constructing, then it must be a T-deck, but the converse may fail. A T-deck that constructs but does not construct a tree, is simply a constructing T-deck.

Note that we allow a card repeated membership in a constructing deck, but not in a proper constructing deck.

Before the following theorem, we would like to relax our definition of cut vertex to include end vertices.

A **snip vertex** v of a graph G is a vertex which is either a cut vertex or an end vertex.

Theorem. For a deck of cards $\mathcal{G} = \{G_i\}$ to be a constructing deck, it is necessary that,

(1) If each card in $\mathcal{G}$ has $p-1$ vertices then there are no more than p cards in $\mathcal{G}$.
(2) For any pair of cards G_i, G_j in $\mathcal{G}$, there are vertices $u_i \in G_i$ and $u_j \in G_j$ such that $G_i - u_i \cong G_j - u_j$.

(3) If a particular card G_i in $\mathcal{G}$ has a component with k vertices, then either G has a component with k vertices, or at least $p-k-1$ of the cards G_j in $\mathcal{G}-G_i$ has a snip vertex v_j such that $G_j - v_j$ has a component with k vertices.

Proof. (1) Clear.

(2) If $\mathcal{G}$ constructs G, then $\mathcal{G}$ is formed by removing each vertex v_i one at a time from G such that $G_i = G - v_i$. Then let $u_j = v_i$ and $u_i = v_j$. The isomorphism is clear since the order in which vertices are removed from a graph does not matter. We get

$$G_i - u_i = (G - v_i) - v_j \cong (G - v_j) - v_i = G_j - u_j.$$

(3) $\mathcal{G}$ is a constructing deck. Say $\mathcal{G}$ constructs G. Without loss of generality, suppose $G_1 = G - v_1$ has a component with k vertices. If G has a component with k vertices we are done. So suppose not. Then G must have a component C_0 with at least $k+1$ vertices since G_1 has a component with k vertices. Thus, v_1 is a snip vertex for C_0 and $C_1 = C_0 - v_1$ has a component with k vertices. Now take $G_j = G - v_j, j \neq 1$. If $v_j \notin C_1$ then v_1 is still a snip vertex for G_j resulting in the component C_1 in $G_j - v_1$. There are k vertices in C_1 and $p-1$ vertices different from v_1 in G. Thus, there are $p-k-1$ vertices not in C_1. Therefore $p-k-1$ cards have a snip vertex (namely v_1) resulting in a component with k vertices.

□

These conditions may or may not be sufficient. We surely can at times tell if a deck is a constructing deck. For instance, the deck,

$$\{P_6, K_1 \cup P_5, K_2 \cup P_4, P_3 \cup P_3, P_4 \cup K_2, P_5 \cup K_1, P_6\}$$

clearly serves to construct P_7. Likewise, we may find certain proper constructing decks. For example, the proper deck $\{K_3\}$ constructs K_4. In fact, we know that $\{K_n\}$ constructs K_{n+1} for $n \geq 3$. Similarly,

$$\{K_p \cup K_{q+1}, K_{p+1} \cup K_q\}$$

constructs $K_{p+1} \cup K_{q+1}$ for $p, q \geq 3$. Actually, Harary's Statements 4 serve to outline some instances in which we know that a deck constructs. Here is a modified version of the Statements 4

Restatements 4. Let $\mathcal{G}$ be a deck. Then,

(1) $\mathcal{G}$ constructs a path if and only if there are exactly two connected cards in $\mathcal{G}$ and they are both paths,
(2) $\mathcal{G}$ constructs a cycle if and only if each card in $\mathcal{G}$ is a path,

(3) $\mathcal{G}$ constructs a complete graph if and only if each card in $\mathcal{G}$ is complete,
(4) $\mathcal{G}$ constructs a totally disconnected graph if and only if each card in $\mathcal{G}$ is totally disconnected.

Corollaries 3. Let $\mathcal{G}$ be a proper deck. Then,

(1) $\mathcal{G}$ constructs C_n iff $\mathcal{G} = \{P_{n-1}\}$,
(2) $\mathcal{G}$ constructs K_n iff $\mathcal{G} = \{K_{n-1}\}$,
(3) $\mathcal{G}$ constructs N_n iff $\mathcal{G} = \{N_{n-1}\}$.

The question remains, however, as to how we could determine if an arbitrary deck is a constructing deck, and even more difficult, if an arbitrary proper deck is a proper constructing deck.

We can in many instances determine when a deck is not constructing. Particularly, if the deck fails any of the necessary conditions we have outlined above.

Less difficult, perhaps, is asking whether a deck in which every card is a forest serves to construct a forest. Certainly if the forest deck serves to construct a tree, then at least two of the cards in the deck must each be a forest with exactly one tree. Is this sufficient to construct a tree? If none of the cards are singular trees, does this suffice to construct a forest? Here we present a small theorem on the matter.

Theorem 2. If a proper T-deck $\mathcal{F}$ T-constructs, then given any pair of T-cards, $T_i, T_j \in \mathcal{F}$, the degree sequences of T_i, T_j differ in exactly two positions and the difference is ± 1 in each position.

Proof. We have a particular T-constructing T-deck $\mathcal{F}$. Say, $\mathcal{F}$ T-constructs the tree T. First, we note that $\mathcal{F}$ does not contain any disconnected cards since they are all trees. Thus $\mathcal{F}$ was obtained from T by snipping only its end-vertices. If $\mathcal{F}$ has only one card then we are done. Thus assume $\mathcal{F}$ has at least two cards. Without loss of generality, consider the cards $T_1 = T - v_1$ and $T_2 = T - v_2$. Clearly v_1 and v_2 are not adjacent in T. Since each of these two vertices are end-vertices of T, they are each adjacent to exactly one other vertex u_1 and u_2 respectively. Note that if $u_1 = u_2$ then $T_1 \cong T_2$ and we would have discarded one of them due to isomorphism. Thus $u_1 \neq u_2$. Now in T_1, $\deg_{T_1} u_1 = \deg_T u_1 - 1$ since we have removed one of its neighbors. And in T_2, $\deg_{T_2} u_1 = \deg_T u_1$, since all of its neighbors are in tact. Thus $\deg_{T_2} u_1 - \deg_{T_1} u_1 = 1$. Clearly, using the same argument, $\deg_{T_2} u_2 - \deg_{T_1} u_2 = -1$. Also clear, all other vertices of T are left unaffected, and the theorem is proved. □

Corollary. With the further condition that for every pair of cards T_i, T_j in a T-deck $\mathcal{F}$ with vertices u_i and u_j where the degree sequences differ, if there is an isomorphism

$$\varphi : V(T_i) - \{u_i\} \longrightarrow V(T_j) - \{u_j\}$$

with $\varphi(v_j) = v_i$, and $\deg v = \deg \varphi(v)$ for all $v \in V(T_i) - \{u_i\}$ then $\mathcal{F}$ T-constructs.

Proof. Take T_1 as above. Add an edge and end vertex to u_1. This is the tree T which $\mathcal{F}$ constructs. $\square$

References

[1] F. Harary. *Theory of Graphs and its Applications*, pages 47 – 52. Academic Press, New York, 1965. On the reconstruction of a graph from a collection of subgraphs.

[2] P. J. Kelly. A congruence theorem for trees. *Pacific Journal of Mathematics*, 7:961 – 968, 1957.