

Reconstructing a Graph from a Collection of Subgraphs

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1. Definitions [1]

Let G be a graph on p vertices and q edges. Then we provide the following definitions.

Definition. A subgraph of G formed by removing exactly one vertex u and edges incident to u (and all labels removed) is called a *card* of G and is written $G - u$.

Definition. The collection of all such subgraphs of G formed by removing exactly one vertex and its edges is the *deck* for G and is denoted by $\text{deck}(G)$.

Definition. A graph H such that $\text{deck}(H) \cong \text{deck}(G)$ is called a *reconstruction* of G .

Definition. G is *reconstructible* if for every reconstruction H of G , we have $H \cong G$.

The Reconstruction Conjecture.

(P.J. Kelly, S.M. Ulam, 1941 [2]). Every finite simple graph on $p \geq 3$ vertices is reconstructible.

2. How to Reconstruct a Graph [1]

2.1. Find the number p of vertices and the number q of edges.

p . Easy; count the number of cards, or count how many vertices any card has and add one.

q . Make a table listing q_i and k_i (the number of components) for $i = 1, 2, \dots, p$. Then

$$q = \frac{1}{p-2} \sum_{i=1}^p q_i.$$

2.2. Decide if G is connected.

Theorem 1. The graph G is connected if and only if at least two of the cards of G are connected.

Corollary. If G is not totally disconnected, then G has at most k components if and only if at least two of the cards of G have at most k components.

2.3. If G is disconnected. Easy. Find the maximal components of G and compare two cards.

2.4. If G is connected.

Optimism. Since $k = 1$, the number m of independent cycles is given by (Kirchhoff):

$$m = q - p + 1.$$

The number b of blocks is given by (Harary):

$$b = 1 + \sum_{i=1}^p (k_i - 1).$$

Connectivity π is given by:

$$\pi = 1 + \min_i \{ \pi_i \}.$$

Miscellaneous (Trivial) Results.

1. G is a path if and only if there are exactly two connected cards of G and they are both paths.
2. G is a cycle if and only if every card of G is a path.
3. G is complete if and only if every card of G is complete.
4. G is totally disconnected if and only if every card of G is totally disconnected.

3. Natural Questions (Assuming the Reconstruction Conjecture Holds)

Question 1. If we remove all cards but one of each isomorphism type, does this *proper deck* reconstruct the graph G ?

Answer. No. Consider $G = K_{1,2}$ and $H = K_1 \cup K_2$. Then G and H have the same proper deck

Question 2. Does the Reconstruction Conjecture hold for directed graphs?

Answer. No. There are many counterexamples.

Question 3. Does the Reconstruction Conjecture hold for infinite graphs? (Here it is probably wise to not carry a full deck).

Answer. Again, no; there are counterexamples.

Question 4. Does the Reconstruction Conjecture hold for general graphs (allowing loops and parallel edges)?

Question 5. What is the minimum number of cards needed to reconstruct a graph?

Answer. The *reconstruction number* of G , $\text{rn}(G)$.

4. A More General form of Question 1.

Question 6. When does a deck \mathcal{G} of cards on $p-1$ vertices construct a graph (uniquely up to isomorphism)?

Definition. A *deck* \mathcal{G} is a collection of graphs with the same number of vertices.

Definition. A deck \mathcal{G} *reconstructs* if there is a unique (up to isomorphism) graph G for which $\mathcal{G} = \text{deck}(G)$.

Question 1 Revised.

When does a proper deck \mathcal{G} reconstruct?

Answer. Sometimes.

Miscellaneous (Trivial) Results Revisited.

Let \mathcal{G} be a deck. Then,

1. \mathcal{G} constructs a path if and only if there are exactly two connected cards in \mathcal{G} and they are both paths,
2. \mathcal{G} constructs a cycle if and only if each card in \mathcal{G} is a path,
3. \mathcal{G} constructs a complete graph if and only if each card in \mathcal{G} is complete,
4. \mathcal{G} constructs a totally disconnected graph if and only if each card in \mathcal{G} is totally disconnected.

Corollaries.

Let \mathcal{G} be a proper deck. Then,

1. \mathcal{G} constructs C_n iff $\mathcal{G} = \{P_{n-1}\}$,
2. \mathcal{G} constructs K_n iff $\mathcal{G} = \{K_{n-1}\}$,
3. \mathcal{G} constructs N_n iff $\mathcal{G} = \{N_{n-1}\}$.

5. Reconstruction Number [3]

For now we will assume that the Reconstruction Conjecture is valid. Then for any graph G on p vertices

Observation 1. $3 \leq \text{rn}(G) \leq p$.

Observation 2. $\text{rn}(\overline{G}) = \text{rn}(G)$.

Theorem 2. If G has 3, 5, or 7 vertices then $\text{rn}(G) = 3$.

Theorem 3. The graph $2K_n$ has reconstruction number $n + 2$ for $n \geq 2$.

Corollary. There are connected graphs with arbitrarily large reconstruction numbers.

5.1. More Conjectures.

Conjecture 1. If G is a graph of odd prime order, then $\text{rn}(G) = 3$.

Conjecture 2. If T is a tree with $p \geq 5$ vertices, then $\text{rn}(T) = 3$.

5.2. A Surprising Result.

Theorem 4. (Bela Bollobàs, 1990 [4]). Almost all (probabilistic sense) graphs have reconstruction number 3.

6. Kelly's Lemma: a Counting Technique [5]

Definition. A *class of graphs* is a family of graphs closed under isomorphism.

Definition. A function defined on a class \mathcal{G} of graphs is *reconstructible* if, for every $G \in \mathcal{G}$, it takes on the same value for all reconstructions of G .

Notation. For graphs G and F , let p_G denote the number of vertices in G ; and let $s_F(G)$ denote the number of subgraphs of G isomorphic to F .

Kelly's Lemma.

(1957). For any two graphs F and G such that $p_F < p_G$, the function $s_F(G)$ is reconstructible.

Corollary. For any two graphs F and G such that $p_F < p_G$, the number of subgraphs of G that are isomorphic to F and contain a given vertex u is reconstructible.

Corollary. The number of edges and the degree sequence are reconstructible.

Corollary. Regular graphs are reconstructible.

7. The Sequel: Edge Reconstruction [5]

The Edge Reconstruction Conjecture.

(F. Harary, 1964). All finite simple graphs on at least four edges are edge reconstructible.

Kelly's Lemma.

(edge version). For any two graphs F and G such that $q_F < q_G$, $s_F(G)$ is edge reconstructible.

Corollary. The number of isolated vertices is edge reconstructible.

Corollary. The Edge Reconstruction Conjecture is valid for all graphs provided that it is valid for all graphs without isolated vertices.

Bibliography

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