A proof of the Heine-Borel Theorem

Theorem (Heine-Borel Theorem). A subset S of $\mathbb{R}$ is compact if and only if S is closed and bounded.

Proof. First we suppose that S is compact. To see that S is bounded is fairly simple: Let $I_n = (-n, n)$. Then

$$\bigcup_{n=1}^{\infty} I_n = \mathbb{R}.$$

Therefore S is covered by the collection of $\{I_n\}$. Hence, since S is compact, finitely many will suffice.

$$S \subseteq (I_{n_1} \cup \cdots \cup I_{n_k}) = I_m,$$

where $m = \max\{n_1, \ldots, n_k\}$. Therefore $|x| \leq m$ for all $x \in S$, and S is bounded.

Now we will show that S is closed. Suppose not. Then there is some point $p \in (\operatorname{cl} S) \setminus S$. For each n, define the neighborhood around p of radius $1/n$, $N_n = N(p, 1/n)$. Take the complement of the closure of N_n, $U_n = \mathbb{R} \setminus \operatorname{cl} N_n$. Then U_n is open (since its complement is closed), and we have

$$\bigcup_{n=1}^{\infty} U_n = \mathbb{R} \setminus \bigcap_{n=1}^{\infty} \operatorname{cl} N_n = \mathbb{R} \setminus \{p\} \supseteq S.$$

Therefore, $\{U_n\}$ is an open cover for S. Since S is compact, there is a finite subcover $\{U_{n_1}, \cdots, U_{n_k}\}$ for S. Furthermore, by the way they are constructed, $U_i \subseteq U_j$ if $i \leq j$. It follows that $S \subseteq U_m$ where $m = \max\{n_1, \ldots, n_k\}$. But then $S \cap N(p, 1/m) = \varnothing$, which contradicts our choice of $p \in (\operatorname{cl} S) \setminus S$.

Conversely, we want to show that if S is closed and bounded, then S is compact. Let $\mathscr{F}$ be an open cover for S. For each $x \in \mathbb{R}$, define the set

$$S_x = S \cap (-\infty, x],$$

and let

$$B = \{x : S_x \text{ is covered by a finite subcover of } \mathscr{F}\}.$$

Since S is closed and bounded, our lemma tells us that S has both a maximum and a minimum. Let $d = \min S$. Then $S_d = \{d\}$ and this is certainly covered by a finite subcover of $\mathscr{F}$. Therefore, $d \in B$ and B is nonempty. If we can show that B is not bounded above, then it will contain a number p greater than $\max S$. But then, $S_p = S$ so we can conclude that S is covered by a finite subcover, and is therefore compact.

Toward this end, suppose that B is bounded above and let $m = \sup B$. We shall show that $m \in S$ and $m \notin S$ both lead to contradictions.

If $m \in S$, then since $\mathscr{F}$ is an open cover of S, there exists F_0 in $\mathscr{F}$ such that $m \in F_0$. Since F_0 is open, there exists an interval $[x_1, x_2]$ in F_0 such that

$$x_1 < m < x_2.$$

Since $x_1 < m$ and $m = supB$, there exists $F_1, \ldots, F_k$ in $\mathscr{F}$ that cover S_{x_1}. But then $F_0, F_1, \ldots, F_k$ cover S_{x_2}, so that $x_2 \in B$. But this contradicts $m = \sup B$.

If $m \notin S$, then since S is closed there exists $\varepsilon > 0$ such that $N(m, \varepsilon) \cap S = \varnothing$. But then

$$S_{m-\varepsilon} = S_{m+\varepsilon}.$$

Since $m - \varepsilon \in B$ we have $m + \epsilon \in B$, which again contradicts $m = \sup B$.

Therefore, either way, if B is bounded above, we get a contradiction. We conclude that B is not bounded above, and S must be compact. □