# A Dictionary for Foundations of Analysis
# (To be expanded by the student)

## Logic

A sentence that can be classified as either true or false is called a **statement**. Let $p$ be a given statement. Then $\sim p$, which we read as "not $p$," is the **negation** of $p$. That is, when $p$ is true, $\sim p$ is false, and vice versa.

A **connective** is a word that connects two statements. Examples include *and*, *or*, *if ... then*, *if and only if.* The connective **and** has the same meaning as in English. In other words, $p$ *and* $q$ is true only when both are true. We will denote this as $p \wedge q$. The connective **or** is used somewhat differently in mathematics than it is often used in English. The statement $p$ *or* $q$ is true when either is true. Note this includes the possibility that both are true. We denote this as $p \vee q$.

A statement of the form "If $p$, then $q$," is called an **implication**. In this statement, the "if $p$" is called the **antecendent** and the "then $q$" part is called the **consequent**. Often, the statement "If $p$, then $q$" is denoted by shorthand "$p \implies q$". If the statement $p \implies q$ appears as a theorem, we will usually call $p$ the **hypothesis** and call $q$ the **conclusion**. The statement "$p$ if and only if $q$" is the combination of the two implications $p \implies q$ and $q \implies p$. This statement is often written as "$p$ iff $q$," or "$p \iff q$," and is called an **equivalence** between the statements $p$ and $q$.

Sometimes we will come across a statement that is true in all cases, such as $p \vee \sim p$. Such a statement is called a **tautology**.

## Quantifiers

The **universal quantifier**, $\forall$, is a symbol which is read as, "For every...," "For each...," "For all...," etc. The **existential quantifier**, $\exists$, is read as, "There exists...," "There is an...," etc. The symbol $\ni$ is shorthand for "...such that..." You need not memorize the names of these symbols, but you should remember what they mean.

## Techniques of Proof

If we have a statement $p \implies q$ whose truth is yet uknown, it takes merely one example where $p$ is true and $q$ is false for the statement $p \implies q$ to be false. Such an example is a **counterexample**.

This suggests a method for proof. Suppose we have the statement $p \implies q$. Then the related statement $\sim q \implies \sim p$ (notice the positions of the $p$ and $q$ in this statement) is equivalent. It is straightforward enough using a truth table to see that the two are equivalent. The statement $\sim q \implies \sim p$ is called the **contrapositive** of $p \implies q$. Often, theorems will be easier to prove when stated in the contrapositive.

The implication $q \implies p$ is called the **converse** of $p \implies q$. It is often the case that $p \implies q$ will be false, while $q \implies p$ will be true. The most familiar example will probably be the relation between squares and rectangles. Another implication related to $p \implies q$ is its **inverse**, $\sim p \implies \sim q$. Like the converse, the inverse is not logically equivalent to the origninal statement $p \implies q$. In fact, the inverse is the contrapositive of the converse.

We mentioned earlier that we will sometimes come across tautologies, statements which are always true. Likewise we will encounter statements which are always false, such as $p \wedge \sim p$. Such a statement is called a **contradiction**. We can often times prove a theorem by supposing that it is false and showing that we arrive logically at a contradiction.

## Set Operations

We are all familiar with the idea of what a collection of objects is. In mathematics, we call a collection of objects a **set**. It is essentially impossible to give a good definition of what a set is without referring to synonyms such as *collection*, *class*, etc. Therefore, mathematicians have agreed that *set* will be exactly what we think it is: a collection of objects whose identity depends only on its members.

You will now see the futility of trying to rigorously define a set: The objects in a particular set are called its **elements** or its **members**. We will usually use capital letters to designate sets, and the symbol $\in$ to indicate membership. Thus, if we write $x \in S$, we mean that the object $x$ is an element of $S$. Conversely, if we write $y \notin S$ we mean that the object $y$ is not a member of $S$.

We can define a particular set in several ways. One method is simply to list the elements of the set. Thus $S = \{3, \pi, a, \text{Saturn}\}$ is the set whose elements are the numbers 3 and $\pi$, the letter $a$, and the planet Saturn. If a set contains a single element, say $b$, then we write $B = \{b\}$. In this way, we have distinguished the element $b$ from the set $\{b\}$ containing $b$ as its only element.

If we are discussing infinite sets, it is obviously not possible to simply list all the elements. Therefore, we will often write down the defining property for the set. For example, the set $P = \{n : n \text{ is prime}\}$ is the set which contains all of the prime numbers.

Okay, now we are ready for some rigorous definitions. Note: when we use "if" in a definition, it is understood to have the same force as "iff".

**Definition.** Let $A$ and $B$ be sets. We say that $A$ is a **subset** of $B$ (or $A$ is **contained** in $B$) if every element of $A$ is an element of $B$. We denote this by $A \subseteq B$ (occasionally, we will write $B \supseteq A$ instead of $A \subseteq B$). If $A \subseteq B$ and $A \neq B$ then $A$ is called a **proper** subset of $B$.

**Definition.** Let $A$ and $B$ be sets. We say that $A$ is **equal** to $B$, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Although we won't yet give formal definitions, we will now name some familiar sets.

- $\mathbb{N}$ will denote the set of positive integers (called the **natural numbers**).
- $\mathbb{Z}$ will denote the set of all integers.
- $\mathbb{Q}$ will denote the set of all rational numbers.
- $\mathbb{R}$ will denote the set of all real numbers.

A very important set in mathematics is the simplest set one can imagine. The **empty set**, denoted $\varnothing$, is the set with no elements. That is, $\varnothing = \{\}$. Note that this is a bona fide set since the statement $x \in \varnothing$ can be classified as true or false for every object $x$. In fact, this statement $x \in \varnothing$ is false for every object $x$.

**Theorem.** Let $A$ be a set. Then $\varnothing \subseteq A$.

*Proof.* To prove that $\varnothing \subseteq A$, we must show that

$$x \in \varnothing \implies x \in A.$$

Since $x \in \varnothing$ is false for all $x$ ($\varnothing$ has no elements), then the implication is always true according to our definition of implies. □

**Definition.** Let $A$ and $B$ be sets. The **union** of $A$ and $B$ (denoted $A \cup B$), the **intersection** of $A$ and $B$ (denoted $A \cap B$), and the **complement** of $B$ in $A$ (denoted $A \setminus B$) are given by

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} \\ A \cap B &= \{x : x \in A \text{ and } x \in B\} \\ A \setminus B &= \{x : x \in A \text{ and } x \notin B\}. \end{aligned}$$

If $A \cap B = \varnothing$, then $A$ and $B$ are said to be **disjoint**.

**Theorem.** Let $A$, $B$, and $C$ be subsets of some universal set $U$ (Here $U$ is simply a set which contains the previous sets). Then the following statements are true:

1. $A \cup (U \setminus A) = U$
2. $A \cap (U \setminus A) = \varnothing$
3. $U \setminus (U \setminus A) = A$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law)
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law)
6. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
7. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

**Definition.** If for each $j$ in a nonempty set $J$ there corresponds a set $A_j$, then

$$\mathscr{A} = \{A_j : j \in J\}$$

is called an **indexed family** of sets with $J$ as the index set. The union of all sets in $\mathscr{A}$ is given by

$$\bigcup\{A_j : j \in J\} = \{x : x \in A_j \text{ for some } j \in J\}.$$

The intersection of all the sets in $\mathscr{A}$ is given by

$$\bigcap\{A_j : j \in J\} = \{x : x \in A_j \text{ for all } j \in J\}.$$

Other notations for $\bigcup\{A_j : j \in J\}$ include

$$\bigcup_{j \in J} A_j \quad \text{and} \quad \bigcup \mathscr{A}.$$

If $J = \{1, 2, ..., N\}$, we may write

$$\bigcup_{j=1}^{n} A_j,$$

and if $J = \mathbb{N}$, we will usually write

$$\bigcup_{j=1}^{\infty} A_j.$$

## Relations

**Definition.** $(a, b) = \{\{a\}, \{a, b\}\}$.

The definition states that the ordered pair $(a, b)$ is a set containing two elements: $\{a\}$ and $\{a, b\}$. Before you object too strongly to this definition, let's see that it has the property that we intuitively attribute to ordered pairs:

**Theorem.** $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

*Proof.* If $a = c$ and $b = d$, then

$$(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d).$$

Conversely, suppose $(a, b) = (c, d)$. Then, by our definition, $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. We wish to show that $a = c$ and $b = d$. There are two cases to consider: $a = b$ and $a \neq b$. We will consider the first case and leave the second to the student.

If $a = b$, then $\{a\} = \{a, b\}$. Thus, $(a, b) = \{\{a\}\}$. Since $(a, b) = (c, d)$, we have

$$\{\{a\}\} = \{\{c\}, \{c, d\}\}.$$

The set on the left has only one element, thus the set on the right must have only one element. Therefore, $\{c\} = \{c, d\}$. We conclude that $c = d$. But then $\{\{a\}\} = \{\{c\}\}$, so $\{a\} = \{c\}$ and $a = c$. □

**Definition.** If $A$ and $B$ are sets, then the **Cartesian product** (or **cross product**) of $A$ and $B$, written $A \times B$, is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. Symbolically,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

**Definition.** Let $A$ and $B$ be sets. A **relation** between $A$ and $B$ is any subset $R$ of $A \times B$. We say that $a \in A$ and $b \in B$ are **related** by $R$ if $(a, b) \in R$, and we will often write this as "$aRb$". If $B = A$, then we speak of a relation $R \subseteq A \times A$ being a relation on $A$.

**Definition.** A relation $R$ on a set $S$ is an **equivalence relation** if it has the following properties for all $x, y, z \in S$:

1. $xRx$ (Reflective Property).
2. If $xRy$, then $yRx$ (Symmetric Property).
3. If $xRy$ and $yRz$, then $xRz$ (Transitive Property).