Math 3210-1 HW 7 Solutions

Due Tuesday, July 27, 2004

Uniform Continuity

19.4 (a) We will prove that if f is uniformly continuous on a bounded set S, then f is bounded on S.

Proof. If S is bounded, then $\operatorname{cl} S$ is also bounded. Since f is uniformly continuous on S, we know that its extension $\widetilde{f}$ must be continuous on $\operatorname{cl} S$. But if $\widetilde{f}$ is continuous on a compact set, then its image must be compact, hence bounded. Therefore, f must be bounded on S. □

(b) By part (a), it is clear that $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1)$ since it is unbounded.

19.8 (a) We wish to show by the Mean Value Theorem that

$$|\sin x - \sin y| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$. We may assume $x < y$. Then sin is continuous on $[x, y]$ and differentiable on (x, y), so we may use the Mean Value Theorem. Hence, there is some point $z \in (x, y)$ such that

$$\sin'(z) = \frac{\sin x - \sin y}{x - y}.$$

Note that $\sin'(z) = \cos z$, apply absolute value to both sides of the above equation, and recall that $|\cos(z)| \leq 1$. Therefore

$$\left| \frac{\sin x - \sin y}{x - y} \right| \leq 1.$$

The result follows. □

(b) We will show that sin is uniformly continuous on $\mathbb{R}$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. We claim that for any $a \in \mathbb{R}$, $0 < |x - a| < \varepsilon$ implies $|\sin x - \sin a| < \varepsilon$. But this follows from part (a) replacing y with a. □

19.10 The only difference between this problem and 19.9 is that $g(x) = x^2 \sin(1/x)$ is **not** uniformly continuous on $\mathbb{R}$, whereas $f(x) = x \sin(1/x)$ is. You can convince yourself of this by noting that the derivative of g is not bounded on $\mathbb{R}$.

The Derivative

28.4 We have $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Clearly f is continuous by the Squeeze Theorem. Also clear is that f is differentiable away from $x = 0$ by the product rule and the chain rule. I will show you part (b). That is, we will see that f is differentiable at $x = 0$. For this we need the definition.

We calculate

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right).$$

This last limit we recognize as having value 0. Thus $f'(0)$ exists and equals zero.

To see that f' is not continuous, we note that away from $x = 0$ we have the formula $f'(x) = 2x \sin(1/x) + \cos(1/x)$. This function does not have a well defined limit at 0 (Why not?), thus has no chance of equalling $f'(0)$.

28.6 f is not differentiable at 0. If we use the definition we get

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x\sin\left(\frac{1}{x}\right)-0}{x}=\lim_{x\to 0}\sin\left(\frac{1}{x}\right).$$

This last limit, we recall, does not exist. Thus, f is not differentiable at $x = 0$. Certainly f is differentiable everywhere else.

28.8 Let $f(x) = x^2$ for $x \in \mathbb{Q}$ and let $f(x) = 0$ for $x \notin \mathbb{Q}$. Then to see that f is continuous at 0 simply apply the Squeeze Theorem, since $0 \le f \le x^2$ for all x. (This is the easy way to do part (a)).

The argument that f is discontinuous for all $x \neq 0$ is similar to what we did in Problem 17.13(b).

To prove that f is differentiable at 0 we **must** use the definition. Consider the following limit:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}.$$

But $f(x)/x = x$ when $x \in \mathbb{Q}$ and $f(x)/x = 0$ when $x \notin \mathbb{Q}$. Thus, this is the familiar function from Problem 17.13(b), which has a limit of 0 at $x = 0$. Hence f is differentiable.

The Mean Value Theorem

29.2 See the solution to Problem 19.8 above.

29.11 To show that $\sin x \le x$ for all $x \ge 0$, first note that $\sin 0 = 0$. Furthermore, if $f(x) = x - \sin x$, then f is an increasing function, since $f'(x) = 1 - \cos(x) \ge 0$.

Taylor's Theorem

31.2

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

32.4 Let $f(x)$ be as in Example 3 from the text.

(a) Take $f_a(x) = f(x-a)$.

(b) Take $g_b(x) = f(b-x)$.

(c) Take $h_{a,b}(x) = f_a(x) \cdot g_b(x)$.

(d) Take $h^*_{a,b}(x) = \dfrac{f_a(x)}{f_a(x)+g_b(x)}$.