

COMPLETENESS AXIOM ,

4.8 Let  $S$  and  $T$  be nonempty subsets of  $\mathbb{R}$  with the following property:  
 $s \leq t$  for all  $s \in S$  and all  $t \in T$ .

(a)  $S$  is bounded above by any  $t_0 \in T$ , and  $T$  is bounded below by any  $s_0 \in S$ . Therefore,  $\sup S$  and  $\inf T$  exist.

(b)  $\sup S \leq \inf T$ .

Proof: Suppose  $\sup S > \inf T$ . Then by the Completeness Axiom,  $\exists x \in \mathbb{R}$  such that  $\sup S > x > \inf T$ . Since  $x < \sup S$ ,  $\exists s_0 \in S$  with  $x < s_0 < \sup S$ . Likewise, since  $x > \inf T$ ,  $\exists t_0 \in T$  with  $x > t_0 > \inf T$ . Thus we get,  $\sup S > s_0 > x > t_0 > \inf T$ .  
In particular,  $s_0 > t_0$ .  $\Downarrow$

(c)  $S = [0, 1]$ ,  $T = [1, 2)$

(d)  $S = [0, 1)$ ,  $T = (1, 2)$ .

## Topology of the Reals

1. (a)  $\emptyset$   
(b)  $(0, 5)$   
(c)  $\emptyset$   
(d)  $\emptyset$   
(e)  $\emptyset$

2. (a)  $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$

(b)  $\{0, 5\}$

(c)  $[0, \sqrt{2}]$

(d)  $[\sqrt{2}, \infty)$

3. (a) neither

(b) closed

(c) neither

(d) This is the empty set in disguise.

Therefore it is both open and closed.

(e) closed

(f) open (this set is  $\mathbb{R} - \{0\}$  in disguise).

4. (a)  $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$

(b)  $\mathbb{N}$

(c)  $\mathbb{R}$

(d)  $\emptyset$

(e)  $[4.5, 5.5]$

(f)  $\mathbb{R}$ .

5. If  $A$  is open and  $B$  is closed, prove that  $A \setminus B$  is open and  $B \setminus A$  is closed.

Proof:

$A \setminus B = A \cap (\mathbb{R} \setminus B)$ , thus is the intersection of two open sets. ■

6. Let  $S$  and  $T$  be subsets of  $\mathbb{R}$ . Then

- (a)  $\text{cl}(\text{cl } S) = \text{cl } S$
- (b)  $\text{cl}(S \cup T) = (\text{cl } S) \cup (\text{cl } T)$
- (c)  $\text{cl}(S \cap T) \subseteq (\text{cl } S) \cap (\text{cl } T)$
- (d) Find an example to show that equality need not hold in part (c)

We have the following items from class at our disposal:

Defn  $\text{cl } S = S \cup S'$ , where  $S'$  is the set of accumulation pts of  $S$ .

- Thm
- (a)  $S$  is closed  $\Leftrightarrow S' \subseteq S$
  - (b)  $\text{cl } S$  is closed
  - (c)  $S$  is closed  $\Leftrightarrow S = \text{cl } S$ .

Now to prove the problem:

Proof: (a) Apply parts (b) and (c) of the theorem to  $S$  and  $\text{cl } S$ .  
 $\text{cl } S$  is closed; and  $\text{cl } S$  is closed  $\Leftrightarrow \text{cl } S = \text{cl}(\text{cl } S)$ . ■

(b)  $(S \cup T)' = S' \cup T'$ , Therefore,

$$\text{cl}(S \cup T) = (S \cup T) \cup (S \cup T)' = (S \cup S') \cup (T \cup T') = (\text{cl } S) \cup (\text{cl } T).$$

(c) While it is not true that  $(S \cap T)' = S' \cap T'$ , it is clear that if  $x \in (S \cap T)'$  then  $x \in S'$  and  $x \in T'$ . Thus

$$(S \cap T)' \subseteq S' \cap T'.$$

Then

$$\begin{aligned} \text{cl}(S \cap T) &= (S \cap T) \cup (S \cap T)' = (S \cup (S \cap T)') \cap (T \cup (S \cap T)') \\ &\subseteq (S \cup (S' \cap T')) \cap (T \cup (S' \cap T')) \subseteq (S \cup S') \cap (T \cup T') = (\text{cl } S) \cap (\text{cl } T). \end{aligned}$$

(d)  $S = (0, 1)$ ,  $T = (1, 2)$ :  $\text{cl}(S \cap T) = \emptyset \neq \{1\} = \text{cl}(S) \cap \text{cl}(T)$ .

7. For any set  $S \subseteq \mathbb{R}$ , denote  $\bar{S} = \bigcap \{K : K \text{ closed and } S \subseteq K\}$ .

(a) Prove that  $\bar{S}$  is closed.

(b) Prove that  $S \subseteq \bar{S}$  and if  $K$  is any closed set containing  $S$ , then  $\bar{S} \subseteq K$ .

(c) Prove that  $\text{cl } S = \bar{S}$ .

(d) If  $S$  is bounded, prove that  $\bar{S}$  is bounded.

proof:  
(a) The intersection of any collection of closed sets is closed.

(b) That  $S \subseteq \bar{S}$  follows from the definition of  $\bar{S}$ .

Now suppose  $C$  is a closed set with  $S \subseteq C$ .

Then if  $x \in \bar{S}$ ,  $x \in \bigcap \{K : K \text{ closed and } S \subseteq K\}$ . In particular,  $x \in C$ . Thus  $\bar{S} \subseteq C$ .

(c) According to the theorem discussed in Exercise 6,  $\text{cl } S$  is a closed set and  $S \subseteq \text{cl } S$ . Thus, by part (b)  $S \subseteq \bar{S} \subseteq \text{cl } S$ . Now take closures.

$\text{cl } S \subseteq \text{cl } \bar{S} \subseteq \text{cl}(\text{cl } S)$ . By the theorem  $\text{cl}(\text{cl } S) = \text{cl } S$ .

By part (a) and the theorem,  $\text{cl } \bar{S} = \bar{S}$ . Thus we get

$\text{cl } S \subseteq \bar{S} \subseteq \text{cl } S$ . We conclude that  $\bar{S} = \text{cl } S$ .

(d) If  $S$  is bounded, then there is a closed bounded set  $K$  with  $S \subseteq K$ . By part (b),  $\bar{S} \subseteq K$ . Thus  $\bar{S}$  is bounded.  $\blacksquare$

## COMPACT SETS

8. (a)  $[1, 3) \subseteq \bigcup_{n=1}^{\infty} (0, 3 - \frac{1}{n})$

(b)  $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} (n - \frac{1}{2}, n + \frac{1}{2})$

(c)  $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, 2)$

you can use any irrational number between 0 and 2.

(d)  $\{x \in \mathbb{Q} : 0 \leq x \leq 2\} \subseteq \bigcup_{n=1}^{\infty} (-1, \frac{\pi}{2} - \frac{1}{n}) \cup (\frac{\pi}{2} + \frac{1}{n}, 3)$

9. Prove that the intersection of any collection of compact sets is compact.

Proof: Such an intersection is bounded by the bounds of any one of the compact sets in the collection. Furthermore the intersection of any collection of closed sets is closed. Therefore, the result follows from Heine-Borel. ■

10. If  $S$  is a compact set of  $\mathbb{R}$  and  $T$  is a closed subset of  $S$ , then  $T$  is compact.

(a) Prove this using the definition of compactness.

(b) Prove this using Heine-Borel.

Proofs: (a) Let  $\{U_{\alpha}\}_{\alpha \in A}$  be an open cover for  $T$ . Since  $T$  is closed,  $\mathbb{R} \setminus T$  is open. Let  $V = \mathbb{R} \setminus T$ , then  $\{V\} \cup \{U_{\alpha}\}_{\alpha \in A}$  is an open cover for  $S$ . Since  $S$  is compact, there is a finite subcover  $\{V\} \cup \{U_{\alpha_k}\}_{k=1, \dots, n}$ . Since  $\{V\}$  covers  $S \setminus T$ , then  $\{U_{\alpha_k}\}_{k=1, \dots, n}$  covers  $T$ . ■

(b)  $S$  is bounded  $\Rightarrow T$  is bounded. ■