Math 3210-1 HW 4

Due Wednesday, July 7, 2004

Problems with only a number listed, such as 1.1, are to be found in *Elementary Analysis*, by Kenneth A. Ross.

Completeness Axiom

Do Exercises 4.1, 4.3, 4.5, 4.7, 4.8, 4.11

Topology of the Reals

1. Find the interior of each set.
   (a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
   (b) $[0, 3] \cup (3, 5)$
   (c) $\{r \in \mathbb{Q} : 0 < r < \sqrt{2}\}$
   (d) $\{r \in \mathbb{Q} : r \geq \sqrt{2}\}$
   (e) $[0, 2] \cap [2, 4]$
2. Find the boundary of each set in Exercise 1, above.
3. Classify each of the following sets as open, closed, neither, or both.
   (a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
   (b) $\mathbb{N}$
   (c) $\mathbb{Q}$
   (d) $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$
   (e) $\left\{x : |x - 5| \leq \frac{1}{2}\right\}$
   (f) $\{x : x^2 > 0\}$
4. Find the closure of each set in Exercise 3, above.
5. If A is open and B is closed, prove that $A \setminus B$ is open and $B \setminus A$ is closed.
6. Let S and T be subsets of $\mathbb{R}$. Prove the following:
   (a) $\mathrm{cl}\,(\mathrm{cl}\, S) = \mathrm{cl}\, S$.
   (b) $\mathrm{cl}\,(S \cup T) = (\mathrm{cl}\, S) \cup (\mathrm{cl}\, T)$
   (c) $\mathrm{cl}\,(S \cap T) \subseteq (\mathrm{cl}\, S) \cap (\mathrm{cl}\, T)$
   (d) Find an example to show that equality need not hold in part (c).
7. For any set $S \subseteq \mathbb{R}$, let $\overline{S}$ denote the intersection of all the closeds sets containing S.
   (a) Prove that $\overline{S}$ is a closed set.
   (b) Prove that $\overline{S}$ is the smallest closed set containing S. That is, show that $S \subseteq \overline{S}$ and if C is any closed set containing S, then $\overline{S} \subset C$.
   (c) Prove that $\overline{S} = \mathrm{cl}\, S$.
   (d) If S is bounded, prove that $\overline{S}$ is bounded.

Compact Sets

8. Show that each of the following subsets of $\mathbb{R}$ is not complact by describing an open cover for it which has no finite subcover.
   (a) $[1, 3)$
   (b) $\mathbb{N}$
   (c) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
   (d) $\{x \in \mathbb{Q} : 0 \leq x \leq 2\}$
9. Prove that the intersection of any collection of compact sets is compact.
10. If S is a compact set of $\mathbb{R}$ and T is a closed subset of S, then T is compact.
    (a) Prove this using the definition of compactness.
    (b) Prove this using the Heine-Borel Theorem.