

Math 3210-1 HW 3

Due Tuesday, June 29, 2004

Problems with only a number listed, such as 1.1, are to be found in *Elementary Analysis*, by Kenneth A. Ross. Problems with a ★ are **very** hard, and should only be attempted after the rest of the assignment is finished.

Cardinality

- Show that each of the following pairs of sets S and T are equinumerous by finding a specific bijection between them.
 - $S = [0, 1]$ and $T = [1, 3]$
 - $S = [0, 1]$ and $T = [0, 1)$
 - $S = [0, 1)$ and $T = (0, 1)$
 - $S = (0, 1)$ and $T = (0, \infty)$
 - $S = (0, 1)$ and $T = \mathbb{R}$
- Prove: If $(S \setminus T) \sim (T \setminus S)$, then $S \sim T$.
- ★ A real number is said to be **algebraic** if it is a root of a polynomial equation

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

with integer coefficients. Note that the algebraic numbers include the rationals and all roots of rationals (such as $\sqrt{2}$, $\sqrt[3]{5}$, etc.). If a number is not algebraic, it is called **transcendental**.

- Show that the set of polynomials with integer coefficients is countable. (Hint: use the fact that the countable union of countable sets is countable).
- Show that the set of algebraic numbers is countable.
- Are there more algebraic numbers or transcendental numbers?

Natural Numbers

Do Exercises 1.1, 1.4, 1.6, 1.9, and 1.11 in Ross.

Rational Numbers

Do Exercises 2.2, 2.3, and 2.6.

Real Numbers

Do Exercises 3.1, 3.3, 3.4, and 3.6. In addition, do the following:

- Prove: If $x \geq 0$ and $x \leq \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.
- Prove that in any ordered field F , $a^2 + 1 > 0$ for all $a \in F$. Conclude from this that if the equation $x^2 + 1 = 0$ has a solution in a field, then that field cannot be ordered. (Thus, it is not possible to define an order relation on the set of all complex numbers that will make it an ordered field.)