MATH 3210 HW2

SOLUTIONS.

1. (a) If there exist violets which are not blue, then there exist roses which are not red.

(b) If G is not normal then G is regular.

(c) If K is not compact, then K is not closed <u>or</u> not bounded.

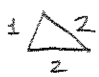
2. (a) If all violets are blue, then all roses are red.

(b) If G is normal, then G is not regular.

(c) If K is compact, then K is closed and bounded.

3. (a) $x = -3$

(b) $n = 41$, then $n^2 + n + 41 = 41(41+1+1)$; ie. not prime.

(c) 

(d) let $n = 101$, or let $n = 2^{24,036,583} - 1$, the 41^{st} known Mersenne Prime.
see http://www.mersenne.org/prime.htm

(e) Let $p = 2$.

(f) Let $n = 1, 3, 5, \ldots$

(g) $x = 0$

(h) $x = -1$.

4. The contrapositive is: If $f(x_1) = f(x_2)$ then $x_1 = x_2$.

<u>proof</u>: Suppose $f(x_1) = f(x_2)$. Then

$$3x_1 - 5 = 3x_2 - 5$$

add 5: $3x_1 = 3x_2$

divide: $x_1 = x_2$.

∎

5. The contrapositive is: If n is odd, then n^2 is odd.

<u>proof</u>: If n is odd, then there exists k so that $n = 2k+1$.

Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2+2k) + 1$.

therefore n^2 is odd.

∎

SET OPERATIONS

6. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

<u>proof</u>: we need to show $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$ and
$(A \setminus B) \cup (B \setminus A) \supseteq (A \cup B) \setminus (A \cap B)$.

we will show "$\subseteq$" first:

Let $x \in (A \setminus B) \cup (B \setminus A)$. Then $x \in (A \setminus B)$ or $x \in (B \setminus A)$.
If $x \in A \setminus B$ then $x \in A$, $x \notin B$. Thus, $x \in A \cup B$ and $x \notin A \cap B$.
Thus $x \in (A \cup B) \setminus (A \cap B)$. Likewise if $x \in (B \setminus A)$ then
$x \in B$, $x \notin A$. Thus $x \in A \cup B$ and $x \notin A \cap B$. Hence we conclude
$x \in (A \cup B) \setminus (A \cap B)$.

Now to show "$\supseteq$".

Let $y \in (A \cup B) \setminus (A \cap B)$. Thus $y \in A$ or $y \in B$, and $y \notin A \cap B$.
If $y \notin A \cap B$ then $y \notin A$ or $y \notin B$. But we know y is in one of them.
Therefore we conclude that either $y \in A$ and $y \notin B$ (i.e. $y \in A \setminus B$),
or $y \in B$ and $y \notin A$ (i.e. $y \in B \setminus A$). Thus, $y \in (A \setminus B) \cup (B \setminus A)$.

∎

7. $A \cap B = A \setminus (A \setminus B)$.

<u>proof</u>: Again, we will prove "$\subseteq$" first.

Let $x \in (A \cap B)$. Then $x \in A$ and $x \in B$. Since $x \in B$, $x \notin A \setminus B$.
Therefore, $x \in A \setminus (A \setminus B)$.

Now we show "$\supseteq$"

Let $y \in A \setminus (A \setminus B)$. Then $y \in A$ and $y \notin (A \setminus B)$. Since $y \notin (A \setminus B)$
then either $y \notin A$ or $y \in B$. But we know $y \in A$. Thus $y \in B$.
Since $y \in A$ and $y \in B$, $y \in A \cap B$.

∎

RELATIONS

8. $A \times B = B \times A$.

This is FALSE. The cartesian product is ordered:

$$A \times B = \{(a,b) : a \in A, b \in B\}.$$

Counterexample:

Let $A = \mathbb{Z}$, $B = \mathbb{R}$.

Then $\mathbb{Z} \times \mathbb{R}$ is the set of ordered pairs whose first entry is an integer and whose second is real, while $\mathbb{R} \times \mathbb{Z}$ has reals in the first slot and integers in the second.

i.e. $(\pi, 1) \in \mathbb{R} \times \mathbb{Z}$,

$(\pi, 1) \notin \mathbb{Z} \times \mathbb{R}$.

9. (a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

proof: First we show "$\subseteq$"

Let $(x,c) \in (A \cup B) \times C$. Then $x \in A \cup B$, $c \in C$. Then either $x \in A$ or $x \in B$. If $x \in A$, then $(x,c) \in A \times C$. If $x \in B$ then $(x,c) \in B \times C$. So $(x,c) \in A \times C$ or $(x,c) \in B \times C$. Thus $(x,c) \in (A \times C) \cup (B \times C)$.

Now to show "$\supseteq$".

Let $(y,d) \in (A \times C) \cup (B \times C)$. Then either $(y,d) \in A \times C$ or $(y,d) \in B \times C$. Either way, $d \in C$. Furthermore, either $y \in A$ or $y \in B$. Thus $y \in A \cup B$. So $(y,d) \in (A \cup B) \times C$. $\blacksquare$

9. (b) $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$.

proof: "$\subseteq$":

Let $(x,y)\in(A\times B)\cap(C\times D)$. Then $(x,y)\in A\times B$ and $(x,y)\in C\times D$. Thus $x\in A$, $y\in B$ and $x\in C$, $y\in D$. Thus $x\in A\cap C$ and $y\in B\cap D$. Thus $(x,y)\in(A\cap C)\times(B\cap D)$.

"$\supseteq$":

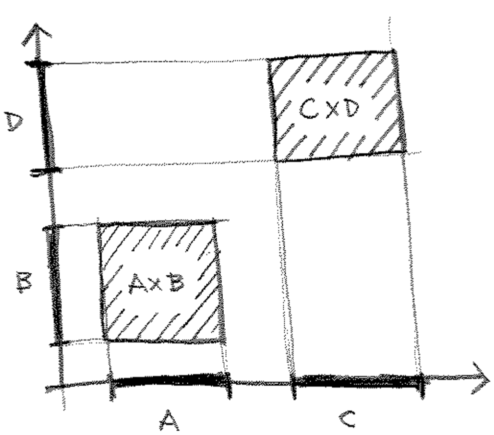
Let $(x,y)\in(A\cap C)\times(B\cap D)$. Then $x\in A\cap C$, $y\in B\cap D$. So $x\in A$, $y\in B$ and $x\in C$, $y\in D$. Thus $(x,y)\in A\times B$ and $(x,y)\in C\times D$. Thus, $(x,y)\in(A\times B)\cap(C\times D)$.

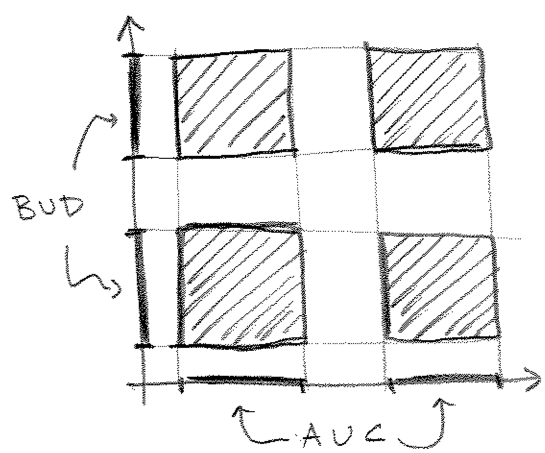
∎

(c) $(A\times B)\cup(C\times D) = (A\cup C)\times(B\cup D)$.

This is FALSE!

consider the following pictures





$(A\times B)\cup(C\times D) \neq (A\cup C)\times(B\cup D)$.

10.

(a) RT

(b) RT

(c) RST

(d) S

(e) ST

(f) RS

11. (a) xRy iff $x < y+2$

There are many many possibilities for these!

(b) xRy iff $x \neq y$

(c) ARB iff $A \subsetneqq B$ for sets A, B.

(d) Define R on lines in the plane where $\ell R k$ iff the lines ℓ and k have at least one point in common.

(e) The relation "$\leq$" defined on $\mathbb{R}$.

(f)

12. $(a,b)R(c,d)$ iff $a=c$.

Reflexive: Certainly $(a,b)R(a,b)$.

Symmetric: If $(a,b)R(c,d)$ then $a=c$, so $c=a$. Thus $(c,d)R(a,b)$.

Transitive: If $(a,b)R(c,d)$ and $(c,d)R(e,f)$ then $a=c$ and $c=e$.

Thus, $a=e$ and $(a,b)R(e,f)$.

The class $E_{(a,b)}$ is all ordered pairs with first entry a.

$$E_{(a,b)} = \{(a,x) : x \in \mathbb{R}\}.$$

geometrically, $E_{(a,b)}$ is the vertical line through (a,b).

13. $S = \mathbb{Z}$, $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m-n \text{ is even}\}$.

ie mRn iff $m-n$ is even.

Reflexive: $\forall m \in \mathbb{Z}$, $m-m=0$ thus mRm.

Symmetric: If mRn then $m-n$ is even. Thus, $n-m$ is even. Hence nRm

Transitive: If mRn and nRp then $m-n$ is even and $n-p$ is even. Thus $m-p$ is even. Therefore mRp.

E_5 is the set of odd integers.

There are only two equivalence classes. Namely, E_5 and E_{-364}.

FUNCTIONS

14. (a) $f(x) = x^2+1$

$\boxed{\text{range}(f) = [1,\infty)}$

(b) $f(x) = (x+3)^2 - 5$

$\boxed{\text{range}(f) = [-5,\infty)}$

(c) $f(x) = x^2+4x+1$

$f'(x) = 2x+4$ $\quad f'(x) = 0$ at $x=-2$

so $f(-2) = -3$ is the minimum.

$\boxed{\text{range}(f) = [-3,\infty)}$.

(d) $f(x) = 2\cos 3x$

$\boxed{\text{range}(f) = [-2,2]}$

15\. (a) f is not injective since there are many (infinitely many) circles with the same area. f is, however, surjective since we can find a circle with any given area in $[0,\infty)$.

(b) Now, since we have prescribed the center point for the circles, g is both injective and surjective.

16\. It may seem there is nothing to prove, but there is: We must show that these functions two functions ($k = h\circ(g\circ f)$ and $l = (h\circ g)\circ f$) take each point $x\in A$ to the same place in D.

proof: We need to show $k(x) = l(x)$ for all $x \in A$.

Let $x\in A$. Define $y=f(x)$ in B, $z=g(y)$ in C, $w=h(z)$ in D.

Now $(g\circ f)(x) = g(f(x)) = g(y) = z$, and

$(h\circ g)(y) = h(g(y)) = h(z) = w$.

Thus,

$$k(x) = h\circ(g\circ f)(x) = h\big((g\circ f)(x)\big) = h(z) = w$$

and

$$l(x) = (h\circ g)\circ f(x) = (h\circ g)(f(x)) = (h\circ g)(y) = w.$$

Therefore, $\forall x\in A$, $k(x)=l(x)$. Thus, $h\circ(g\circ f) = (h\circ g)\circ f$.

17\. $f:\mathbb{N}\to\mathbb{N}$

(a) surjective, not injective: $f(n) = \lceil n/2 \rceil$ (which we read as the "ceiling" of $n/2$).

$\lceil n/2 \rceil$ means, divide n by 2 and round up when n is odd.

$$\begin{aligned} 1 &\mapsto 1\\ 2 &\mapsto 1\\ 3 &\mapsto 2\\ 4 &\mapsto 2\\ 5 &\mapsto 3\\ 6 &\mapsto 3\\ 7 &\mapsto 4\\ &\vdots \end{aligned}$$

7. (b) injective, not surjective: $f(n) = p_n$, the n^{th} prime.

$$\begin{aligned}
1 &\mapsto 2\\
2 &\mapsto 3\\
3 &\mapsto 5\\
4 &\mapsto 7\\
5 &\mapsto 11\\
6 &\mapsto 13\\
7 &\mapsto 17\\
&\vdots
\end{aligned}$$

(c) neither surjective nor injective: $f(n) = 31$. The constant function.

d) bijective: $f(n) = n$, the identity function, is the most obvious. Here is another:

$$\begin{aligned}
1 &\to 2\\
2 &\to 1\\
3 &\to 5\\
4 &\to 3\\
5 &\to 4\\
6 &\to 9\\
7 &\to 6\\
8 &\to 7\\
9 &\to 8\\
&\vdots
\end{aligned}$$

Do you see the pattern. Note there need not be a pattern so long as each number is "hit" exactly once.

18. $f: A \longrightarrow B$, $C \subseteq A$, $D \subseteq B$.

(a) $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$.

Proof: There are two implications to prove. We will first prove "$\Longrightarrow$".

Suppose $f(C) \subseteq D$. Therefore, if $x \in C$ then $f(x) \in D$.

But $f^{-1}(D) = \{x \in A : f(x) \in D\}$. Thus, $C \subseteq f^{-1}(D)$.

Now to show "$\Longleftarrow$".

Suppose $C \subseteq f^{-1}(D)$. That is, if $x \in C$, then $f(x) \in D$.

Let $y \in f(C)$. Then there exists $z \in C$ so that $y = f(z)$.

But $z \in C$ implies $f(z) \in D$. Thus $y \in D$, and we conclude

$f(C) \subseteq D$.

(b) For $f(C) = D$ iff $C = f^{-1}(D)$ we require that

D be in the image of f, that is $D \subseteq f(A)$.

19. Let $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = e^x$ and

$g : \mathbb{R} \to \mathbb{R}$ via $g(x) = x^2$

Then $(g \circ f)(x) = e^{2x}$ is injective, but g is not injective.

20. $g : A \longrightarrow C$, $h : B \longleftrightarrow C$ is bijective, then $\exists\, f : A \to B$ so that $g = h \circ f$.

proof: This is the picture we want.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed, "f"'] & C \arrow[d, leftrightarrow, "h"] \\ & B \end{tikzcd}$$

want f so that $g = h \circ f$.

Since h is bijective, for every $y \in C$ there is exactly on element $z \in B$ so that $h(z) = y$. Therefore, we can declare $h^{-1}(y) = z$. (we say the inverse is "well-defined").

Now we will define the function $f : A \to B$.

Let $x \in A$. Then $g(x) \in C$. By what we said above, there exists a unique $h^{-1}(g(x)) \in B$ so that $h(h^{-1}(g(x))) = g(x)$.

Let $f(x) = h^{-1}(g(x))$. Then we conclude that $g = h \circ f$.