Math 2280-1 Final Exam

Thursday, August 4, 2005.

SOLUTIONS

You may use pen or pencil, scratch paper, and two sheets (front and back, $8\frac{1}{2} \times 11$) of notes. Calculators are **not** allowed (or needed) for this exam. There are 125 points possible on this exam. The value of each problem is marked. Show all of your work. Good Luck!

1. (5 points) Are the functions $f(x) = 7, g(x) = \sin^2 x$, and $h(x) = \cos^2 x$ linearly independent? Prove your assertion. Hint: Do NOT use the Wronskian, that's the long way! (If you can't think of any other way, tell me exactly how you would use the Wronskian and I'll give you partial credit.)

No $\quad f(x) - 7g(x) - 7h(x) = 0$.

2. (5 points) Solve

$$(2x\cos(x^2) + y) + x\frac{dy}{dx} = 0, \quad y(\sqrt{2\pi}) = 0.$$

(Hint: you can use integrating factors, but that is the HARD way).

Let $M = 2x\cos(x^2) + y$

$N = x$

Then $\quad M_y = 1 = N_x$ so the equ. is exact!

Integrate.

$$F(x,y) = \int M\,dx = \int 2x\cos(x^2) + y\,dx$$

$$= \sin x^2 + xy + g(y).$$

$$F(x,y) = \int N\,dy = \int x\,dy = xy + h(x).$$

Thus

$$F(x,y) = \sin x^2 + xy + C = 0.$$

But since $y(\sqrt{2\pi}) = 0$, $C = 0$.

$$\sin x^2 + xy = 0$$

$$y = -\frac{\sin x^2}{x}.$$

3. (5 points) Use a substitution to transform the second-order equation

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3$$

into a first order equation. Do NOT solve this equation!

Let $p = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p$.

$$\Longrightarrow \quad y\, p \frac{dp}{dy} = p^3.$$

4. (5 points) Rewrite the homogenous linear differential equation

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + x = 0.$$

as a system of first order equations. (Note that this problem is asking something quite different from what is being asked in Problem 3.)

Let

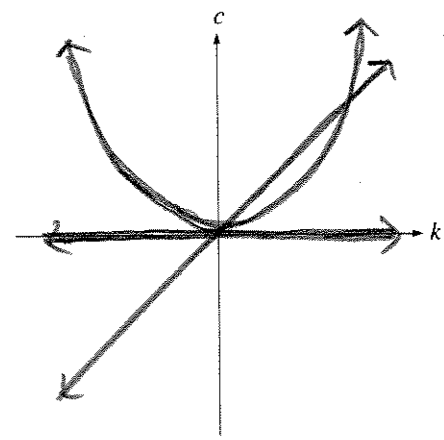
$$\boxed{\begin{aligned} \dot{x} &= y \\ \ddot{x} = \dot{y} &= z \\ \dddot{x} = \dot{z} &= -2z + 3y - x \end{aligned}}$$

5. (15 points) Consider the differential equation

$$\frac{dx}{dt} = -Ax(x-k)(x-k^2),$$

where k is a parameter and A is a positive constant.

(a) Draw a bifurcation diagram for the parameter k.



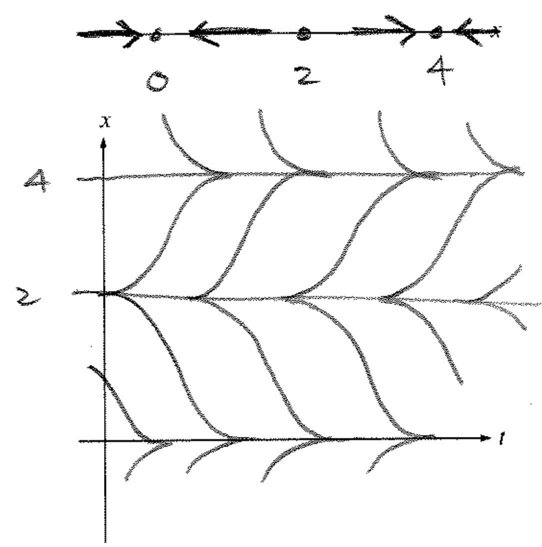
(b) Fix $k = 2$. Draw a phase diagram, then draw solution curves for this system.

Fixed points:

$x = 0$,

$x = 2$

$x = 4$

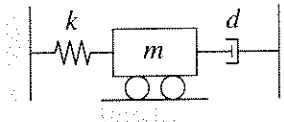


(c) Now suppose the above equation models a population, and $k = 10$. ~~Use your results from part (a) to~~ interpret the physical meaning of the populations $x = 10$ and $x = 100$.

$x = 10$ is the THRESHOLD POPULATION for growth.

$x = 100$ is the CARRYING CAPACITY for the environment.

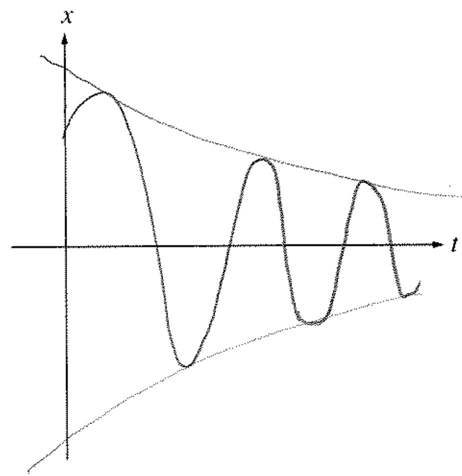
6. (10 points) How can you tell whether the mechanical system below is Underdamped, Overdamped, or Critically Damped? What is the behavior of the system in each of these cases given some nontrivial initial conditions? Draw some typical solution curves for each type of system.



Define $\zeta = 2\sqrt{km}$.

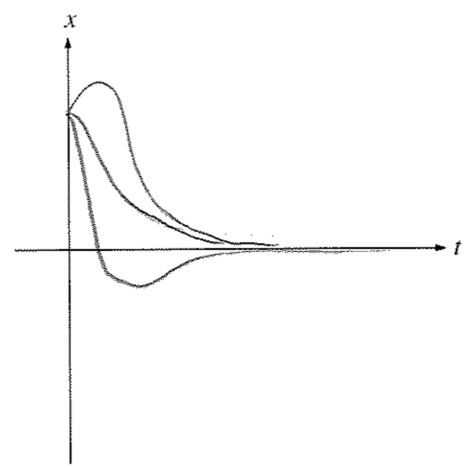
Underdamped:

$d < \zeta$



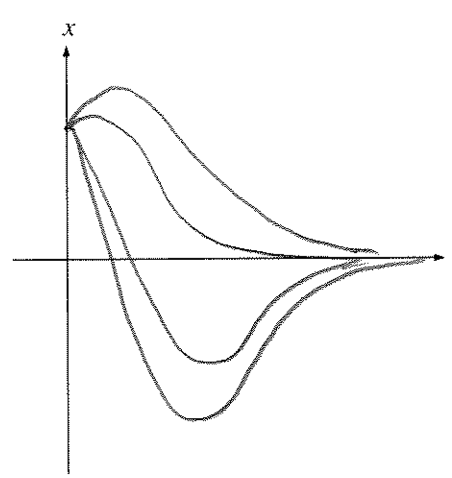
Overdamped:

$d > \zeta$



Critically Damped:

$d = \zeta$



7. (10 points) Consider the nonhomogeneous differential equation,

$$\frac{d^2y}{dx^2} + y = \tan x.$$

(a) Show that $y_1(x) = \cos x$ and $y_2(x) = \sin x$ solve the associated homogeneous equation.

(b) Show that the Wronskian of y_1 and y_2 is $W(y_1, y_2)(x) = 1$.

(c) Compute the general solution to the nonhomogenous equation through Variation of Parameters. You may use the formula directly if you know it. You may leave any integrals you encounter unevaluated.

(a) $\frac{d^2y_1}{dx^2} = -\cos x = -y_1.$ Thus $y_1'' + y_1 = 0$

$\frac{d^2y_2}{dx^2} = -\sin x = -y.$ Thus $y_2'' + y_2 = 0.$

(b) $W(y_1, y_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

(c)

$y(x) = y_1 \int \frac{y_2 f}{W} + y_2 \int \frac{y_1 f}{W}$

$y(x) = \cos x \int \frac{\sin x \tan x}{1} dx + \sin x \int \frac{\cos x \tan x}{1} dx$

$= \cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x \, dx.$

$\square$

8. (10 points) The homogeneous linear differential equation

$$\frac{d^4y}{dx^4} + \omega^2 \frac{d^2y}{dx^2} = 0,$$

can be used to model a uniform column buckling under axial load (where ω^2 is a parameter measuring material properties of the column, and the load to which it is subjected).

(a) Check that $y_1(x) = \sin \omega x$, $y_2(x) = \cos \omega x$, $y_3(x) = x$, and $y_4(x) = 1$ are all solutions to this differential equation (just substitute). Thus the general solution to the DE is

$$y(x) = c_1 \sin \omega x + c_2 \cos \omega x + c_3 x + c_4.$$

(b) Now consider the boundary conditions $y(0) = y''(0) = 0$. What does this tell us about the constants c_2 and c_4?

(c) Now consider the boundary conditions $y(L) = y''(L) = 0$. What does this tell us?

(d) Which values of ω yield a nontrivial solution to this boundary value problem? What are the characteristic solutions associated to these values?

(a) $y_1^{(4)} = w^4 \sin wx$

$y_1^{(2)} = -w^2 \sin wx$. Thus $y_1^{(4)} + w^2 y_1^{(2)} = 0$.

Similarly, $y_2^{(4)} + w^2 y_2^{(2)} = 0$.

$y_3^{(4)} = 0$ $\quad$ $y_4^{(4)} = 0$

$y_3^{(2)} = 0$. $\quad$ $y_4^{(2)} = 0$.

(b) $y(0) = 0 \Rightarrow c_2 + c_4 = 0$

$y''(0) = 0 \Rightarrow c_2 = 0 \Rightarrow c_4 = 0$.

(c) $y(L) = 0 \Rightarrow c_1 \sin wL + c_3 L = 0$

$y''(L) = 0 \Rightarrow c_1 \sin wL = 0 \Rightarrow c_3 = 0$.

But $c_1 \sin wL = 0 \Rightarrow$ either $c_1 = 0$ or

$\sin wL = 0 \left(\Rightarrow w = \frac{n\pi}{L} \right)$.

(d) We get nontrivial solutions for $c_1 \neq 0 \Rightarrow$ $w_n = \frac{n\pi}{L}$. The associated solutions are

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right).$$

9. (10 points) Let $\mathbf{B} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$.

(a) Verify that $\mathbf{B}^2 = \mathbf{B}$. From this, one can prove that $\mathbf{B}^n = \mathbf{B}$ for all $n \in \mathbb{N}$ (You may do so for extra credit; use induction).

(b) Use the result that $\mathbf{B}^n = \mathbf{B}$ to show that $\exp(\mathbf{B}) = \mathbf{I} + (e-1)\mathbf{B}$. Therefore,

$$\exp(\mathbf{B}) = \begin{bmatrix} 1 & 3e-3 \\ 0 & e \end{bmatrix}.$$

(a) $B^2 = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$.

Suppose, par induction, that $B^{n-1} = B$.

Then $B^n = B^{n-1} \cdot B = B \cdot B = B^2 = B$.

(b)

$$\exp(B) = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \frac{1}{4!}B^4 + \cdots$$

$$= I + B + \frac{1}{2!}B + \frac{1}{3!}B + \frac{1}{4!}B + \cdots$$

$$= I + B\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$

$$= I + B(e-1)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3e-3 \\ 0 & e-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3e-3 \\ 0 & e \end{bmatrix}.$$

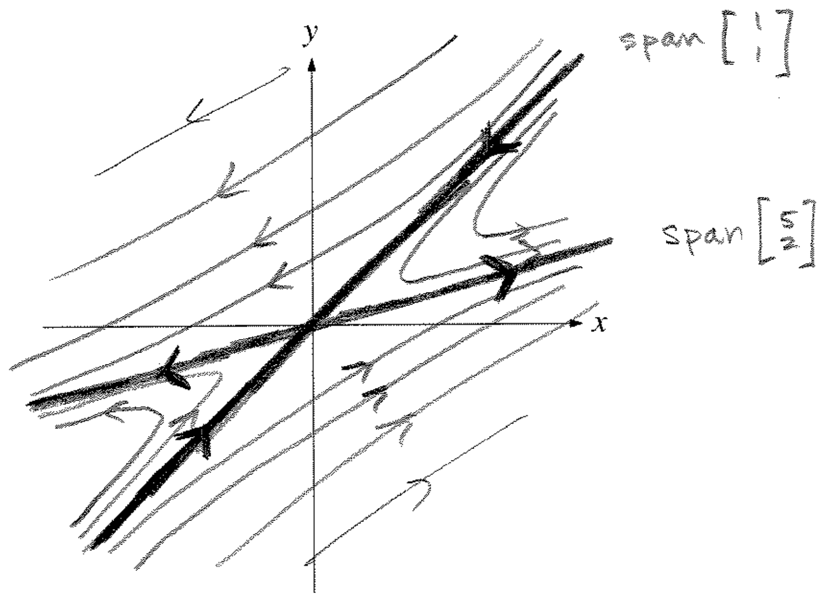
10. (15 points) Let $\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 2 & -4 \end{bmatrix}$.

(a) Use $\det(\mathbf{A})$ and $\operatorname{trace}(\mathbf{A})$ to compute the eigenvalues: $\lambda_{1,2} = -2, 1$.

(b) Verify that the associated eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

(c) Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Based on the eigenvalues, what is the nature of the critical point at the origin?

(d) On the axes below, sketch the solution curves given by the eigenvectors. Then, sketch other trajectories near the origin. Be sure to put arrows on your curves to indicate the direction of the trajectory.



(e) Write down the general solution to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

(a) $\det A = -2$ $\quad \operatorname{Tr} A = -1$. Thus the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1$. Since $\det A = \lambda_1 \cdot \lambda_2$ and $\operatorname{trace} A = \lambda_1 + \lambda_2$.

(b)

$$\begin{bmatrix} 3 & -5 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -5 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

(c) Since they are real with opposite signs, it is a SADDLE.

(e) $$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{t}.$$

11. (15 points) Consider the system

$$\frac{dx}{dt} = x^3 - x$$
$$\frac{dy}{dt} = x + y$$

(a) Find all critical points of the system (there are three, and one is the origin).

(b) Choose one of the critical points away from the origin. Linearize the system at that point.

(c) What type of critical point is it. You should be able to use the determinant and trace to compute the eigenvalues easily enough.

(a) crit points.

$f(x,y) = x^3 - x$.

has zeros at $x = 0$, $x = \pm 1$.

$g(x,y) = x + y$.

when $x = 0$, $y = 0$

when $x = \pm 1$, $y = \mp 1$.

$$\boxed{(0,0)\ ;\ (1,-1)\ ;\ (-1,1)}$$

(b) $J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 - 1 & 0 \\ 1 & 1 \end{bmatrix}$.

At $(1,-1)$, $J = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$.

At $(-1,1)$, $J = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

I made these the same on purpose
☺ Easier to grade.

The linear system in either case is

$\dot{x} = 2x$
$\dot{y} = x + y$.

(c) The eigenvalues are 2, 1, so it is an improper nodal source.

12. (10 points) Suppose that you have a nonlinear autonomous system

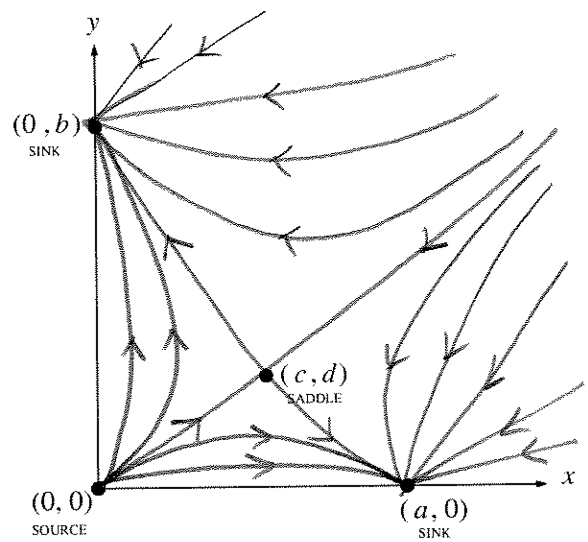
$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

that models two competing species, and this system has the following critical points in the first quadrant:

- $(0,0)$ is a proper source;
- $(a,0)$ is a proper sink;
- $(0,b)$ is a proper sink;
- (c,d) is a saddle;

where a, b, c, and d are all positive constants.

(a) Sketch probable trajectories for this nonlinear system; be sure to connect critical points appropriately, and to draw arrows to indicate direction.



(b) What is the ultimate fate of this ecology? (Can both species coexist peacefully? What role do the initial conditions play? etc.) Be as inclusive as possible in your analysis.

Unless the ICs are EXACTLY on the curve through (0,0) and (c,d), the ecology moves towards the extinction of one population in favor of the other. There cannot be peaceful coexistence! The curve through these points is called the seperatrix. If the ICs are above this curve, population Y will survive. If the ICs are below, population X will. In either case, the surviving population tends toward CARRYING CAPACITY.

13. (10 points) Solve the initial value problem

$$\ddot{x} + 2\dot{x} = 2\delta(t), \quad x(0) = 0, \dot{x}(0) = 0.$$

Laplace transform :

$$s^2 X(s) + 2sX(s) = 2 \quad (\text{since } x(0) = \dot{x}(0) = 0).$$

$$X(s) = \frac{2}{s(s+2)}.$$

Inverse Laplace Transform :

$$x(t) = \int_0^t 2e^{-2\tau} \, d\tau$$

$$= -e^{-2\tau} \Big|_{\tau=0}^{t}$$

$$= 1 - e^{-2t}.$$