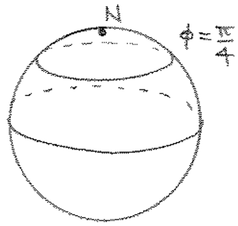
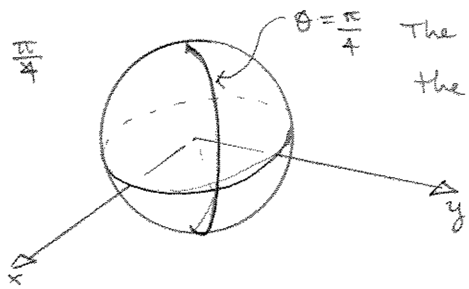
PARAMETERIZED SURFACES

154 (a) $\phi = \frac{\pi}{4}$:



The circle $\frac{\pi}{4}$ radians from the north pole.

(b) $\theta = \frac{\pi}{4}$



The semicircle $\frac{\pi}{4}$ radians from the x-axis.

155. (a) $x =$

$y =$

$z =$

(b) $x = 5\sin\phi\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = 5\sin\phi\sin\theta$, $\quad 0 \le \phi \le \pi$.

$z = 5\cos\phi$

(c) $x = 2 + 5\sin\phi\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = -1 + 5\sin\phi\sin\theta$ $\quad 0 \le \phi \le \pi$

$z = 3 + 5\cos\phi$

(d) $x = R\sin\phi\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = R\sin\phi\sin\theta$ $\quad 0 \le \phi \le \pi$

$z = R\cos\phi$

(e) $x = t\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = t\sin\theta$

$z = t$

156.

$x = a \sin\phi \cos\theta \qquad 0 \le \theta \le 2\pi$

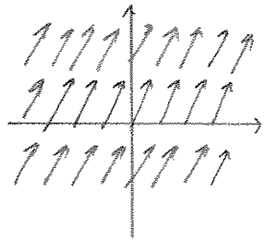
$y = b \sin\phi \sin\theta \qquad 0 \le \phi \le \pi$

$z = c \cos\phi$

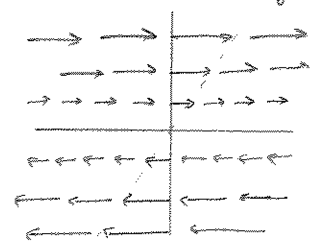
VECTOR FIELDS

161.

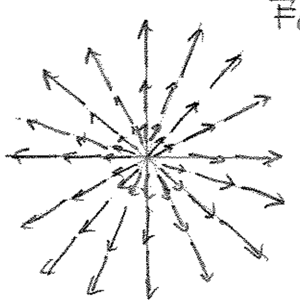
(a) $\vec{F} = \langle 2,3 \rangle$



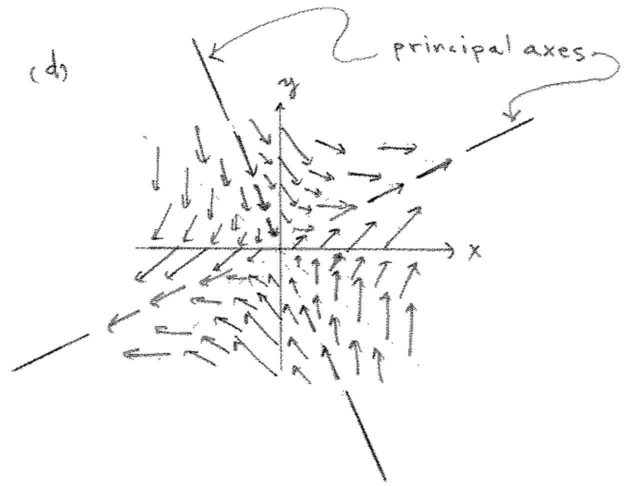
(b) $\vec{F} = y\hat{\imath}$



(c) $\vec{F}(\vec{r}) = 2\vec{r}$



(d)

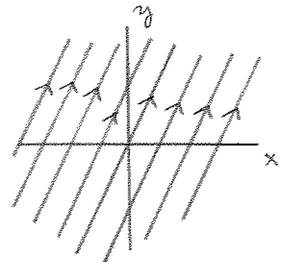


162 See next page.

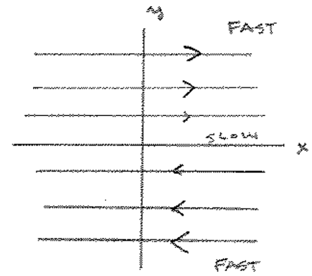
THE FLOW OF A VECTOR FIELD.

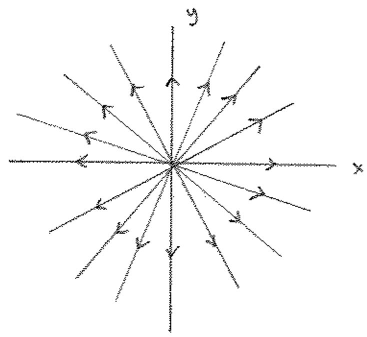
163.

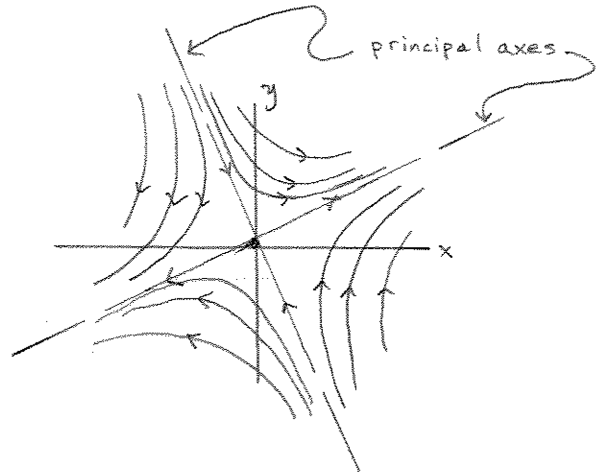
(a)



(b)







164

(a) $\frac{dx}{dt} = y$ $\qquad x = a(e^t + e^{-t}) \qquad y = a(e^t - e^{-t})$

$\frac{dy}{dt} = x$ $\qquad \frac{dx}{dt} = a(e^t - e^{-t}) = y$

$\frac{dy}{dt} = a(e^t + e^{-t}) = x$ ✓

(b) $\frac{dx}{dt} = y$ $\qquad x = a\sin t \qquad y = a\cos t$

$\frac{dy}{dt} = -x$ $\qquad \frac{dx}{dt} = a\cos t = y \qquad \frac{dy}{dt} = -a\sin t = -x$ ✓

(c) $\frac{dx}{dt} = x$ $\qquad x = ae^t \qquad y = be^t$

$\frac{dy}{dt} = y$ $\qquad \frac{dx}{dt} = ae^t = x \qquad \frac{dy}{dt} = be^t = y$ ✓

(d) $\frac{dx}{dt} = x$ $\qquad x = ae^t \qquad y = be^{-t}$

$\frac{dy}{dt} = -y$ $\qquad \frac{dx}{dt} = ae^t = x \qquad \frac{dy}{dt} = -be^{-t} = -y$ ✓

165.

$$\frac{dr}{dt} = ar - crf \qquad \frac{df}{dt} = -bf + krf.$$

The solution to these DEs is the flow to the vector field:

$$\vec{V} = \langle ar - crf, -bf + krf \rangle.$$

THE IDEA OF A LINE INTEGRAL

166.

$$\int_{C_3} \vec{F} \cdot dr \le \int_{C_1} \vec{F} \cdot \vec{dr} \le \int_{C_2} \vec{F} \cdot dr$$

167.

(a) positive

(b) zero

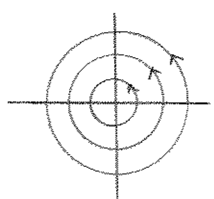
(c) positive

(d) negative

168.

(a) $\vec{F}(\vec{r}) = \vec{r}$. zero

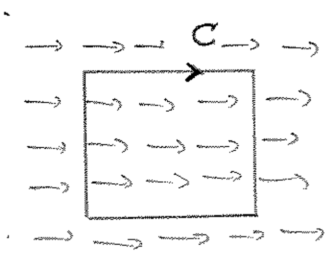
(b) $\vec{F} = \langle -y, x \rangle$ positive



(c) $\vec{F} = \langle y, -x \rangle$ negative

(d) $\vec{F} = x^2 \hat{\imath}$ zero

169.



$\vec{F} = \hat{\imath}$

MANY possible examples.

COMPUTING LINE $\int$s

170.

(a) $\vec{F} = \langle x^2, y^2 \rangle$

C: $x = 1+t$ $\quad 0 \le t \le 2$
$y = 2+t$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt}\, dt = \int_0^2 \langle 1+2t+t^2, 2+4t+t^2 \rangle \cdot \langle 1, 1 \rangle\, dt$$

$$= \int_0^2 3 + 6t + 2t^2\, dt = 3t + 3t^2 + \frac{2}{3}t^3 \Big|_0^2$$

$$= 6 + 12 + \frac{16}{3} = \boxed{\frac{70}{3}}$$

(b) $\vec{F} = \langle \ln y, \ln x \rangle$; C: $\vec{r}(t) = \langle 2t, t^3 \rangle \quad 2 \le t \le 4$.

(This is the cubic $y = \frac{1}{8}x^3$ from $(4,8)$ to $(8,64)$).

$$\int_C \vec{F} \cdot d\vec{r} = \int_2^4 \langle \ln t^3, \ln 2t \rangle \cdot \langle 2, 3t^2 \rangle\, dt$$

$$= \int_2^4 6 \ln t + 3t^2 \ln 2t \quad dt$$

$$= \int_2^4 6 \ln t + 3t^2 \ln 2 + 3t^2 \ln t \; dt \quad \text{(use Table of } \textstyle\int\text{s)}$$

$$= 6\left[t \ln t - t\right]\Big|_2^4 + t^3 \ln 2 \Big|_2^4 + 3\left[\frac{t^3}{9}(3 \ln t - 1)\right]\Big|_2^4$$

$$= 6\Big(4\ln 4 - 4 - \underbrace{2 \ln 2}_{\ln 4} + 2\Big) + (64-8)\ln 2 + \frac{1}{3}\Big(64\big(\underbrace{3 \ln 4}_{6 \ln 2} - 1\big) - 8(3 \ln 2 - 1)\Big)$$

$$= 6(3 \ln 4 - 2) + 56 \ln 2 + \frac{1}{3}\Big(64 \cdot 6 \ln 2 - 4 \cdot 6 \ln 2 - 64 + 8\Big)$$

$$= 36 \ln 2 - 12 + 56 \ln 2 + 120 \ln 2 - \frac{56}{3} = \boxed{212 \ln 2 - \frac{92}{3}} \approx 116.28\ldots$$

170 (c) $\vec{F} = \langle e^x, e^y \rangle$ C: $x = 2\sin t$, $y = \cos t$ $0 \le t \le \pi/2$

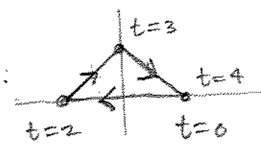
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle e^{2\sin t}, e^{\cos t} \rangle \cdot \langle 2\cos t, -\sin t \rangle \, dt$$

$$= \int_0^{\pi/2} \left[e^{2\sin t} \, 2\cos t + e^{\cos t}(-\sin t) \right] dt$$ u-sub.

$u = 2\sin t$, $du = 2\cos t \, dt$ $u = \cos t$, $du = -\sin t \, dt$

$$= \int_0^1 e^u \, du + \int_1^0 e^u \, du = \boxed{0}$$

(d) $\vec{F} = \langle xy, x-y \rangle$ C :



C_1 : $x = 1 - t$, $y = 0$ $0 \le t \le 2$

C_2 : $x = -1 + t$, $y = t$ $0 \le t \le 1$

C_3 : $x = t$, $y = 1 - t$ $0 \le t \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \langle 0, 1-t \rangle \cdot \langle -1, 0 \rangle \, dt + \int_0^1 \langle (t-1)(t), -1 \rangle \cdot \langle 1, 1 \rangle \, dt + \int_0^1 \langle t(1-t), 2t-1 \rangle \cdot \langle 1, -1 \rangle \, dt$$

$$= 0 + \int_0^1 \cancel{t^2} - \cancel{t} - \cancel{1} \, dt + \int_0^1 \cancel{t} - \cancel{t^2} - 2t + \cancel{1} \, dt$$ combine!

$$= \int_0^1 -2t \, dt = -t^2 \Big|_0^1 = \boxed{-1}$$

170. (e) $\vec{F} = \langle x, 2zy, x \rangle$ $\quad C: \vec{r}(t) = \langle t, t^2, t^3 \rangle \quad 1 \le t \le 2$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 \langle t, 2t^3t^2, t \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_1^2 t + 4t^6 + 3t^3 \, dt$$

$$= \frac{1}{2}t^2 + \frac{4}{7}t^7 + \frac{3}{4}t^4 \Big|_1^2$$

$$= 2 + \frac{4}{7}(128) + \frac{3}{4}(16) - \frac{1}{2} - \frac{4}{7} - \frac{3}{4}$$

$$= \frac{3}{2} + \frac{4}{7}(127) + \frac{3}{4}(15) = \frac{6}{4} + \frac{45}{4} + \frac{508}{7}$$

$$= \frac{51}{4} + \frac{508}{7} = \boxed{\frac{2389}{28}} \approx 85.32\ldots$$

171. $\vec{F} = \langle -y, x \rangle$. $\quad C: x^2 + y^2 = 1$. or $x = \cos t$, $y = \sin t$

(a) $\|\vec{F}\| = (-y)^2 + x^2 = y^2 + x^2 = 1$.

(b) The velocity vector for C is $\frac{d\vec{r}}{dt} = \langle -\sin t, \cos t \rangle$

$$= \langle -y, x \rangle$$

$$= \vec{F}.$$

so $\vec{F}$ is tangent to C.

(c) $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \cdot dt$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} dt = 2\pi = \text{length}(C).$$

(d) $\|\vec{F}\| = 1$ and $\vec{F} \parallel d\vec{r}$ Thus $\vec{F} \cdot d\vec{r} = \|d\vec{r}\| = dr$.

so $\int_C \vec{F} \cdot d\vec{r} = \int_C dr$ = total length along C = length(C).

(General Principle of Integration).