The Derivative.

30. Let $s(t) = 5t^2$

(a) $s(0) = 0$, $s(1) = 5$. So $s(1) - s(0) = \boxed{5 \text{ meters.}}$

(b) $s(2) = 20$. So $s(2) - s(1) = \boxed{15 \text{ meters.}}$

(c) $s(4) = 80\,m$, $s(5) = 125\,m$.

$$v_{avg} = \frac{125m - 80m}{1 sec} = \boxed{45 m/sec,}$$

(d) $s(4.01) = 80.4005$

$$v_{avg} = \frac{80.4005m - 80m}{0.01 sec.} = \boxed{40.05 m/s}$$

(e)

$$v_{inst.} = \lim_{h\to 0} \frac{5(4+h)^2 = 5(4)^2}{h}$$

$$= \lim_{h\to 0} \frac{5(16+8h+h^2) - 5(16)}{h}$$

$$= \lim_{h\to 0} \frac{40h + 5h^2}{h} = \lim_{h\to 0} 40 + 5h = \boxed{40 m/s}.$$

31. (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \left(\frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}\right) = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) $f(x) = \frac{3}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \lim_{h\to 0} \frac{\frac{3x - 3(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}x(x+h)} = -\frac{3}{x^2}.$$

31 (c) $f(x) = \dfrac{x}{x+1}$

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x+1) - x(x+h+1)}{(x+1)(x+h+1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{x^2+(h+1)x+h - x^2 - xh - x}{(x+1)(x+h+1)}$$

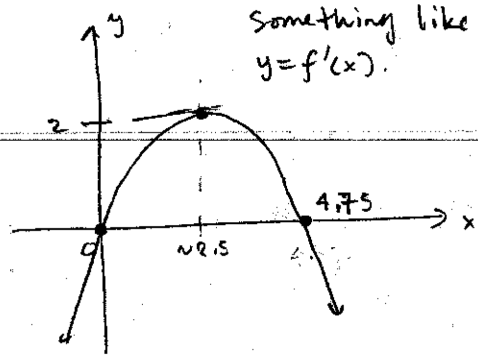
$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \frac{\cancel{h}}{(x+1)(x+h+1)}$$

$$= \lim_{h\to 0} \frac{1}{(x+1)(x+h+1)} = \boxed{\frac{1}{(x+1)^2}} = \frac{1}{x^2+2x+1}$$

32. (a) $f'(0) = 0$
$f'(2) \simeq 2$
$f'(4) \simeq 1$
$f'(6) \simeq -\frac{3}{2}$

(b) $x = 0$ and $x \simeq 4.75$
$f'(x) = 0$ at these points.

(c) Something like this.
$y = f'(x)$.



33.  $= f(x)$

 $= f'(x)$

34. $r_{ave.} = \dfrac{800\text{ kgal} - 100\text{ kgal}}{24\text{ hrs.}}$

$\cong \boxed{29\text{ kgal/hr}}$

(It is positive!)

$r_{8am} \simeq \dfrac{200\text{ kgal}}{4\text{ hrs.}}$

$= \boxed{50\text{ kgal/hr}}$

Rules for Finding Derivatives

35. Many examples possible.
must take functions to functions (or numbers)
and must be linear:

$$L(af+g) = aLf + Lg.$$

36. (a) $D_x y = 6x$

(b) $D_x y = k$

(c) $D_x y = 9x^8 + 7x^6 + 5x^4 + 3x^2 + 1$

(d) $D_x y = -\frac{1}{x^2} + 2x$

(e) $D_x y = -3\pi x^{-4}$

(f)
$$D_x y = (2x+1)(x-1) + (x^2+x+1)\,1$$
$$= 2x^2 - 2x + x - 1 + x^2 + x + 1$$
$$= 3x^2.$$

Easy way:
$$y = (x^2+x+1)(x-1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1.$$
$$D_x y = 3x^2.$$

(g)
$$D_x y = \frac{1\,(x^2+x+1) - (2x+1)(x-1)}{(x-1)^2}$$
$$= \frac{x^2+x+1 - 2x^2+2x-x+1}{(x-1)^2}$$
$$= \frac{-x^2+2x+2}{(x-1)^2}$$

(h)
$$y = 17\,\frac{x^4+x^3}{x^2-x+1}$$

$$D_x y = 17\,\frac{(4x^3+3x^2)(x^2-x+1) - (x^4+x^3)(2x-1)}{(x^2-x+1)^2}$$

You can stop here.

$$= 17\,\frac{4x^5 - 4x^4 + 4x^3 + 3x^4 - 3x^3 + 3x^2 - 2x^5 + x^4 - 2x^4 + x^3}{(x^2-x+1)^2}$$

$$= 17\,\frac{2x^5 - 2x^4 + 2x^3 + 3x^2}{(x^2-x+1)^2} = \boxed{17x^2\,\frac{2x^3-2x^2+2x+3}{(x^2-x+1)^2}}$$

36. (i) $\boxed{D_x y = -\frac{6}{23} x^{-3}}$

(j) $\boxed{D_x y = -10x^{-6} - 9x^{-4} - x^{-2}}$

(k) $D_x y = \boxed{987{,}654{,}321\, x^{987654320}}$

(l) $y = (x-4)^2 = x^2 - 8x + 16$

$D_x y = 2x - 8 = \boxed{2(x-4)}$

(m) $y = (x^2+1)^3 \underset{\uparrow}{=} (x^2)^3 + 3(x^2)^2 + 3(x^2)^1 + 1 = x^6 + 3x^4 + 3x^2 + 1$

Binomial Theorem

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$\boxed{D_x y = 6x^5 + 12x^3 + 6x}$

$= 6x(x^4 + 2x^2 + 1)$

$= 6x(x^2+1)^2.$

37 (a) $D_x[f(x)]^2 = D_x[f(x) \cdot f(x)] =$

$= D_x f(x) \cdot f(x) + f(x) \cdot D_x f(x)$

$= 2f(x) D_x f(x)$ ∎

(b) $D_x[f(x)]^3 = D_x[f(x) \cdot (f(x))^2]$

$= D_x f(x) \cdot (f(x))^2 + f(x) \cdot (2f(x) D_x f(x))$

$= 3(f(x))^2 D_x f(x).$ ∎

(c) $D_x[f(x)]^n = n(f(x))^{n-1} D_x f(x).$

38. (a) $(f+g)'(1) = f'(1) + g'(1) = 2 + 4 = \boxed{6}$

(b) $(f \cdot g)'(1) \neq f'(1) g'(1)$!!!

$(f \cdot g)'(1) = f'(1) g(1) + f(1) g'(1) = 2 \cdot 3 + 1 \cdot 4 = \boxed{10}$

(c) $(f/g)'(1) = \dfrac{f'(1) g(1) - f(1) g'(1)}{(g(1))^2} = \dfrac{2 \cdot 3 - 1 \cdot 4}{3^2} = \boxed{\dfrac{2}{9}}$

39. Let $F(x) = f(x) - g(x)$.

$$F'(x) = \lim_{h \to 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) - (g(x+h) - g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x) \quad \blacksquare$$

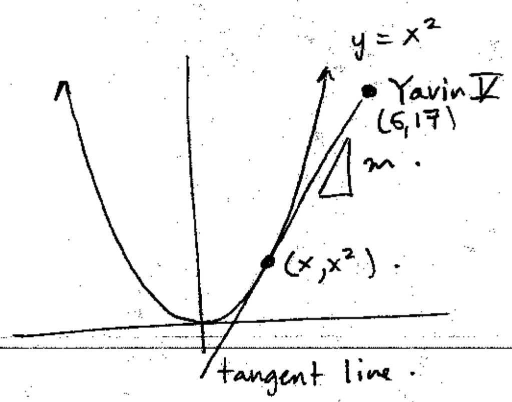
40. We want $y' = 0$.

$$y' = \tfrac{1}{3} \cdot 3x^2 + \tfrac{1}{2} \cdot 2x + 0$$

$$= x^2 + x.$$

$$x^2 + x = 0 \iff x(x+1) = 0 \iff \boxed{x = 0 \text{ or } x = -1}$$

41.



We need to compute the slope in two ways:

(i) $m = \frac{\text{rise}}{\text{run}} = \frac{17 - x^2}{5 - x}$

(ii) $m = y' = 2x$.

$$\implies 2x = m = \frac{17 - x^2}{5 - x} \iff 2x(5 - x) = 17 - x^2$$

$$\iff 10x - 2x^2 = 17 - x^2$$

$$x^2 - 10x + 17 = 0.$$

Quadratic Formula:

$$x = \frac{10 \pm \sqrt{100 - 4(17)}}{2}$$

$$= \frac{10 \pm \sqrt{32}}{2} = \frac{10 \pm 4\sqrt{2}}{2}$$

$$= 5 \pm \sqrt{2}.$$

If Chewie is traveling from left to right, he should cut power at $5 - \sqrt{2}$.
If going the other way, $5 + \sqrt{2}$

42. $y_1 = x^2$ $\qquad y_2 = -x^2 + \frac{3}{2}x + 1$

$m_1 = y_1' = 2x$ $\qquad m_2 = y_2' = -2x + \frac{3}{2}$.

For the tangent lines to be $\perp$, we need

$$m_2 = -\frac{1}{m_1}.$$

$$\Longrightarrow \quad -2x + \frac{3}{2} = -\frac{1}{x}.$$

$$\Longrightarrow \quad x\left(2x - \frac{3}{2}\right) = 1$$

$$2x^2 - \frac{3}{2}x - 1 = 0$$

$$4x^2 - 3x - 2 = 0$$

Use Quadratic Formula:

$$x = \frac{3 \pm \sqrt{9 - 4(4)(-2)}}{2(4)}$$

$$= \frac{3 \pm \sqrt{9 + 32}}{8}$$

$$= \boxed{\frac{3 \pm \sqrt{41}}{8}}.$$

Derivatives of Trig Functions

43. (a) $D_x y = \boxed{3\cos x - 2\sin x}$

(b) $D_x y = D_x(\sin x \cdot \sin x \cdot \sin x)$

$= \cos x \cdot \sin^2 x + \cos x \cdot \sin^2 x + \cos x \cdot \sin^2 x$

$= \boxed{3\sin^2 x \cos x}.$

(c) $D_x y = \boxed{\sec^2 x + \cos x}.$

(d) $D_x y = (\cos x)(\cos x) + (\sin x)(-\sin x)$

$= \boxed{\cos^2 x - \sin^2 x}.$

(e) $$D_x y = \frac{(\cos x + \sin x)(\sin x - \cos x) - (\cos x + \sin x)(\sin x + \cos x)}{(\sin x - \cos x)^2}$$

any of three

$$= -1 - \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^2$$

$$= -1 - \left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)^2$$

Note that this is $D_x y = -1 - y^2$.

(f) $\boxed{D_x y = 1 \cdot \sin x + x\cos x}$

(g) $\boxed{D_x y = 3x^2 \cos x + x^3 \sin x}$

(h) Let $f(x) = x\sin x + \cos x$, $g(x) = x^2 + 1$

$f'(x) = \sin x + x\cos x - \sin x \qquad g'(x) = 2x.$

$= x\cos x$

Then $$D_x y = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{x\cos x\,(x^2+1) - 2x(x\sin x + \cos x)}{(x^2+1)^2} = \boxed{\frac{x^3\cos x - x\cos x - 2x^2\sin x}{(x^2+1)^2}}$$

(i) $y = 1$. So $\boxed{D_x y = 0}$ (j) $\boxed{D_x y = -\sin x - 1}$

44. $f(x) = \cos(3x)$

Use
$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

$$f'(x) = \lim_{h\to 0} \frac{\cos(3x+3h) - \cos(3x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(3x)\cos(3h) - \sin(3x)\sin(3h) - \cos(3x)}{h}.$$

$$= \lim_{h\to 0} -\cos(3x)\frac{(1-\cos(3h))}{h} - \sin(3x)\frac{\sin(3h)}{h}$$

$$= \lim_{h\to 0} -3\cos(3x)\frac{(1-\cos(3h))}{3h} - 3\sin(3x)\frac{\sin(3h)}{3h}$$

Let $k = 3h$
$k \to 0$ as $h \to 0$.

$$= \lim_{k\to 0} -3\cos(3x)\frac{1-\cos k}{k} - 3\sin 3x \frac{\sin k}{k}$$

$$= -3\cos(3x)\cdot 0 - 3\sin(3x)\cdot 1$$

$$= \boxed{-3\sin 3x}$$

45. $y = \cos x$.
$y(\pi/2) = \cos(\pi/2) = 0$. So $(\pi/2, 0)$ is on our line.

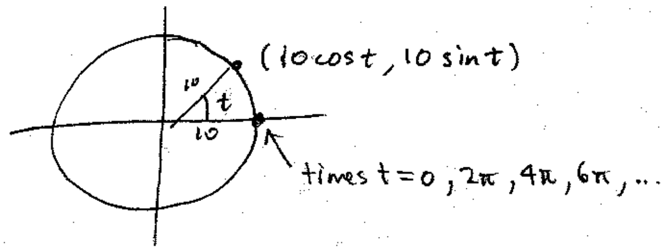
$y' = -\sin x$.
$y'(\pi/2) = -\sin(\pi/2) = -1$.

So our slope $m = 1$.
Using Point-Slope Form: $y - y_1 = m(x - x_1)$

$$y - 0 = 1(x - \pi/2)$$

$$\boxed{y = x - \pi/2}$$

46.



(a) $x = 10 \cos t$ meters

$y = 10 \sin t$ meters

(b) $\frac{dy}{dt} = 10 \cos t$ m/s

(c) It is rising most quickly at

$t = 0$ sec, 2π sec, 4π sec, ...

Its vertical speed at these times is 10 m/s.