Introduction to Limits

1. (a) $\lim\limits_{x \to 2} (x-3)$

when x is near 2, $x-3$ is near $2-3=-1$

So

$$\lim_{x \to 2} (x-3) = -1 .$$

(b) $\lim\limits_{x \to -1} (x^2+3x-4)$

when x is near -1, x^2-3x-4 is near $(-1)^2+3(-1)-4 = -6$

So

$$\lim_{x \to -1} (x^2+3x-4) = -6$$

(c) $\lim\limits_{x \to -3} \dfrac{x^2-4x-21}{x+3}$.

$\dfrac{x^2-4x-21}{x+3}$ is not defined AT $x=-3$, but that is not what is being asked. We need to know what $\dfrac{x^2-4x-21}{x+3}$ is <u>close to</u> when x is <u>close to</u> -3.

First some algebra.

$$\frac{x^2-4x-21}{x+3} = \frac{(x+3)(x-7)}{x+3} = x-7, \; x \neq -3 .$$

So, when x is close to -3, $\dfrac{x^2-4x-21}{x+3}$ is close to $(-3)-7=-10$.

$$\lim_{x \to -3} \frac{x^2-4x-21}{x+3} = -10 .$$

(d) $$\lim_{y \to 0} \frac{y^5+2y^4+y^3}{y^3} = \lim_{y \to 0} y^2+2y+1 = 1 .$$

(e) $$\lim_{t \to -5} \frac{\sqrt{(t+5)^3}}{t+5} = \lim_{t \to -5} \frac{(t+5)^{3/2}}{t+5} = \lim_{t \to -5} (t+5)^{1/2} = \lim_{t \to -5} \sqrt{t+5}$$

So when t is close to -5, $\dfrac{\sqrt{(t+5)^3}}{t+5}$ is close to $\sqrt{(-5)+5} = 0$

$$\lim_{t \to -5} \frac{\sqrt{(t+5)^3}}{t+5} = 0 .$$

2.

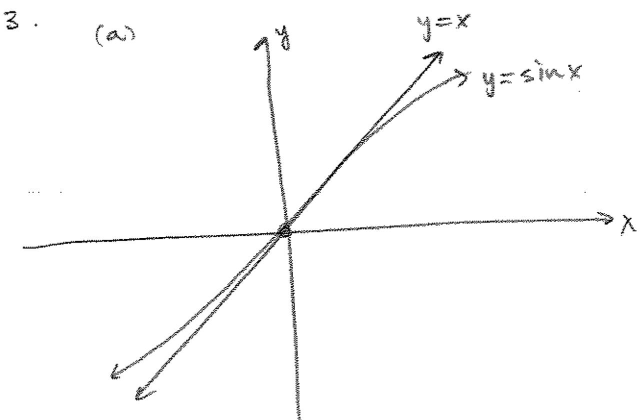
x	(a) $\frac{\sin x}{x}$	(b) $\frac{1-\cos x}{x}$
1		
0.5		
0.1		
0.01		
0.001		
0.0001		
0.00001		
0.000 001		
0.000 0001		

Guess:

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(b) $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$.

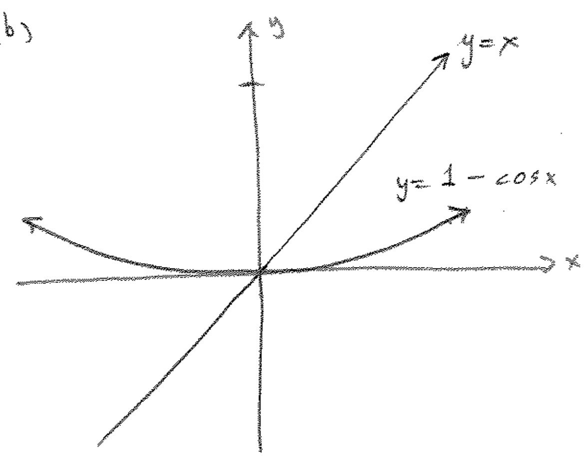
3. (a)



$y=x$ and $y=\sin x$ go to zero at the same speed.

$\therefore \frac{\sin x}{x} \to 1$ as $x \to 0$

(b)



$y = 1-\cos x$ goes to zero much faster than $y=x$ goes to zero.

$\therefore \frac{1-\cos x}{x} \to 0$

as $x \to 0$.

4. (a) $\lim_{x \to -2} f(x) = 1.5$

(b) $f(-2) = 3$

(c) $\lim_{x \to -3} f(x)$ does not exist

(d) $f(-3) = 1$

(e) $\lim_{x \to 1} f(x)$ does not exist

(f) $f(1) = 2$

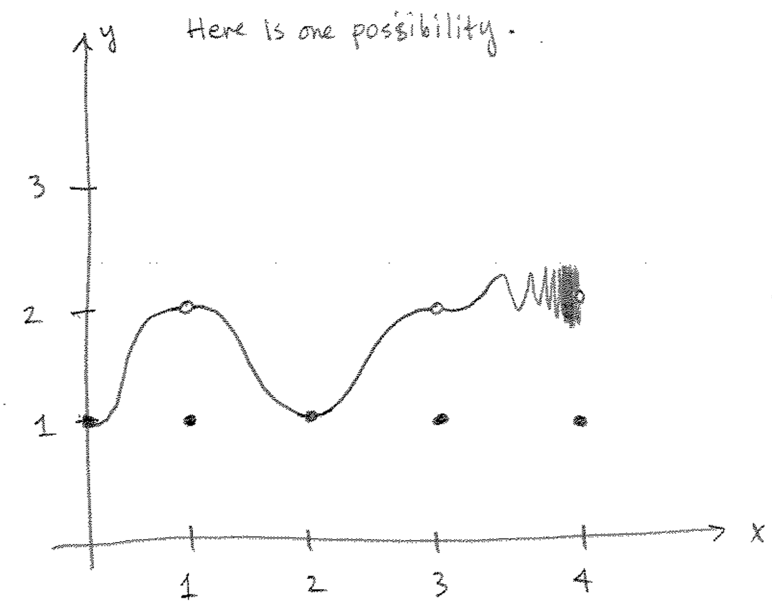
(g) $\lim_{x \to 2} f(x) = 2.5$

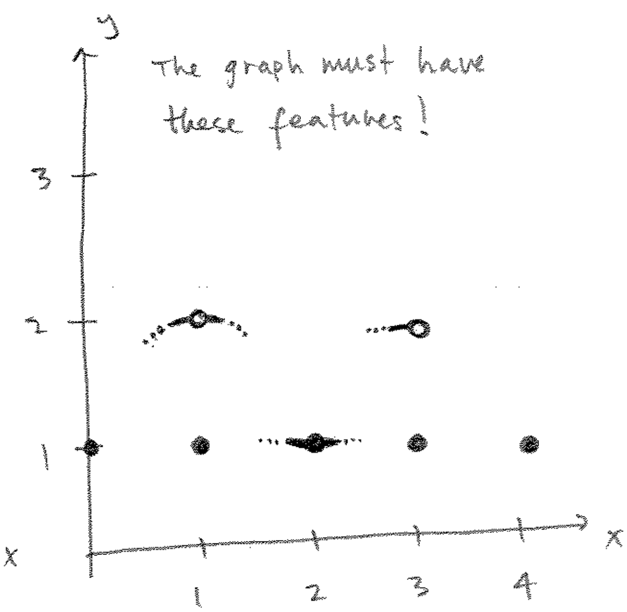
(h) $f(2)$ does not exist.

(i) $\lim_{x \to 1^+} f(x) = 1$ (Limit from the right).

(j) $\lim_{x \to -2^-} f(x) = 1.5$ (Limit from the left).

5.





6. Changing Finitely many points will never change any limits.

Rigorous Study of Limits.

7. Skipped.

8. Skipped.

Limit Theorems

9. (a) $\lim_{x \to 1} (3x+2) \underset{\uparrow \text{MLT}}{=} 3\left(\lim_{x\to 1} x\right) + 2 = 3(1) + 2 = 5$

(b) $\lim_{x \to 0} (2x+5)(x+1) \underset{\uparrow \text{substitution theorem}}{=} (5)(1) = 5$

(c) $\lim_{x \to 2} \frac{x+1}{x-1} = \frac{3}{1} = 3.$

(d) $\lim_{x \to -1} \sqrt{x^2+3} \underset{\uparrow \text{MLT}}{=} \sqrt{\lim_{x\to -1}(x^2+3)} \overset{\text{substitution}}{=} \sqrt{4} = 2$

(e) $\lim_{t \to 2} \left(\frac{7t+3}{t^2+t+1}\right)^{1/5} \overset{\text{MLT}}{=} \left(\lim_{t\to 2} \frac{7t+3}{t^2+t+1}\right)^{1/5} \overset{\text{substitution}}{=} \left(\frac{17}{7}\right)^{1/5}$.

10. $\lim_{x\to a} f(x) = 1 \quad \lim_{x \to a} g(x) = 2$

(a) $\lim_{x\to a} f(x) - g(x) = 1 - 2 = -1$

(b) $\lim_{x\to a} (f(x))^2 + (g(x))^2 = 5$

(c) $\lim_{x\to a} \frac{f(x)-g(x)}{f(x)+g(x)} = \frac{-1}{3}$

(d) $\lim_{x\to a} \left(g^2(x) - f^2(x)\right)^4 = 3^4 = 81.$

11. Many possible solutions. Here's one: $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x}$.
$\lim_{x\to 0} f(x) + g(x) = 0$, but $\lim_{x\to 0} f(x)$ does not exist.

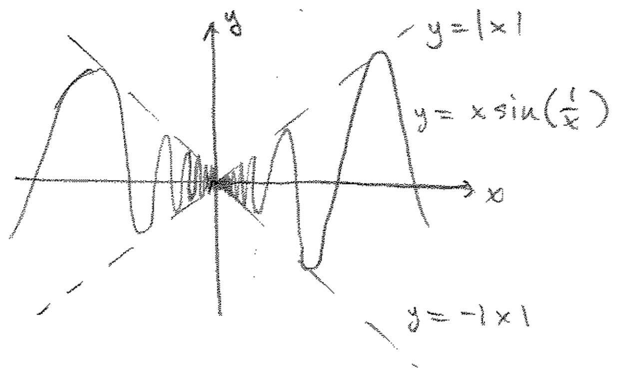
12. Many possible solutions. Here's one: $f(x) = \frac{1}{x}$, $g(x) = x^2$.
$\lim_{x\to 0} f(x)\cdot g(x) = \lim_{x\to 0} \frac{x^2}{x} = 0$, but $\lim_{x\to 0} f(x)$ does not exist.

13. $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

$\lim\limits_{x\to 0} -|x| = \lim\limits_{x\to 0} |x| = 0$

So, by the Squeeze Theorem,

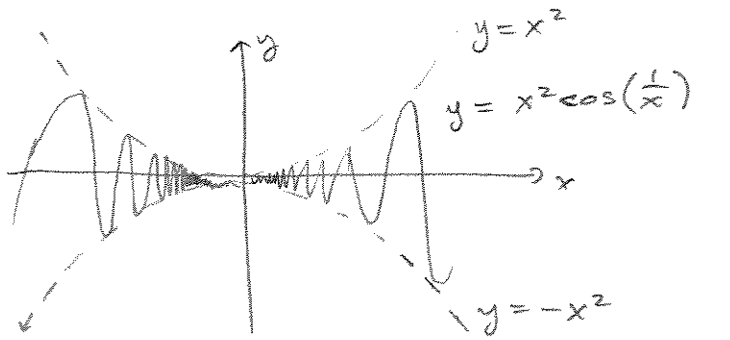
$x\sin\left(\frac{1}{x}\right) \to 0$ also.



14. $-x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2$

as $x \to 0$, $x^2 \to 0$ and $-x^2 \to 0$,

So $x^2\cos\left(\frac{1}{x}\right) \to 0$ also.



Trigonometric Limits

15. (a) $\lim\limits_{t\to 0} \dfrac{\sin t}{1+t} = \dfrac{\sin 0}{1+0} = \dfrac{0}{1} = 0.$

(b) $\lim\limits_{t\to 0} t\sin t = 0\cdot\sin 0 = 0.$

(c) $\lim\limits_{t\to 0} \dfrac{\sin^2 t}{t} = \lim\limits_{t\to 0} \sin t\cdot\dfrac{\sin t}{t}$

$= \left(\lim\limits_{t\to 0}\sin t\right)\left(\lim\limits_{t\to 0}\dfrac{\sin t}{t}\right) = 0\cdot 1 = 0.$

(d) $\lim\limits_{t\to 0} \dfrac{1-\cos^2 t}{t} = \lim\limits_{t\to 0} \dfrac{\sin^2 t}{t} = 0$, by part (c).

Because $\cos^2 t + \sin^2 t = 1$.

(e) $\lim\limits_{t\to 0} \dfrac{\sin^2 t}{t(1+\cos t)} =$

$= \lim\limits_{t\to 0} \dfrac{\sin^2 t}{t(1+\cos t)}\cdot\dfrac{(1-\cos t)}{1-\cos t} = \lim\limits_{t\to 0} \dfrac{\sin^2 t\,(1-\cos t)}{t\,(1-\cos^2 t)}$

$= \lim\limits_{t\to 0} \dfrac{\cancel{\sin^2 t}(1-\cos t)}{t\,\cancel{\sin^2 t}} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{t} = 0.$

Asymptotic Limits & Infinite Limits

16. (a) $\lim_{x \to \infty} \frac{x}{x+2} = 1$

(b) $\lim_{x \to \infty} \frac{x^2}{x^3+2} = \lim_{x \to \infty} \frac{x^2 \cdot \left(\frac{1}{x^2}\right)}{(x^3+2)\left(\frac{1}{x^2}\right)} = \lim_{x \to \infty} \frac{1}{x + \frac{2}{x^2}} = 0.$

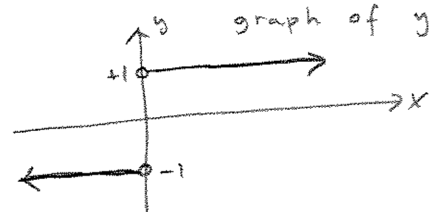
(c) $\lim_{x \to \infty} \frac{x^3}{x^2+2x+1} = \infty$

(d) $\lim_{x \to \infty} \frac{\sin\theta}{\theta^2+1} = 0$

(e) $\lim_{x \to -2^+} \frac{x}{x+2} = -\infty$ because the numerator is close to -2, and the denominator is close to 0^+, meaning it is small, but still positive.

* Note: $\lim_{x \to -2^-} \frac{x}{x+2} = +\infty$ because the denominator is now close to 0^-.

graph of $y = \frac{|x|}{x}$



(f) $\lim_{x \to 0^+} \frac{|x|}{x} = 1.$

* Note $\lim_{x \to 0^-} \frac{|x|}{x} = -1.$

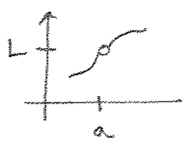
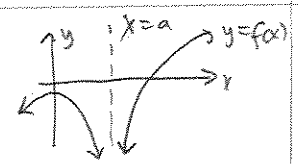
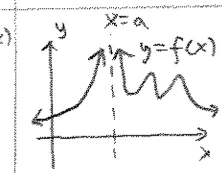
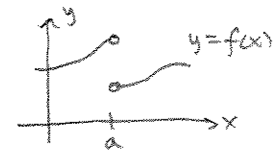
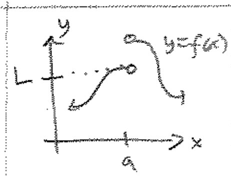
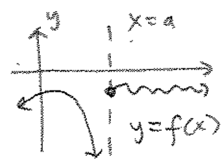
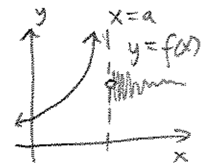
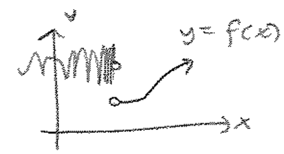
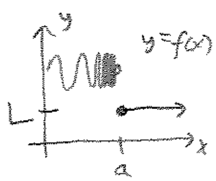
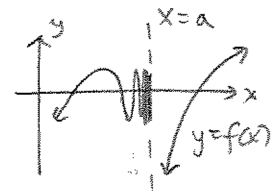
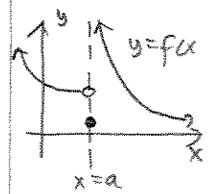
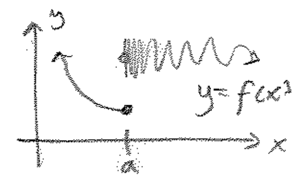
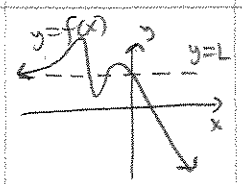
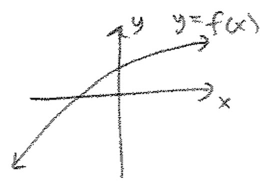
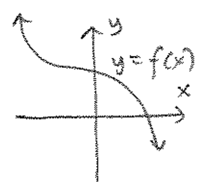
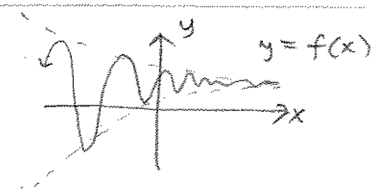
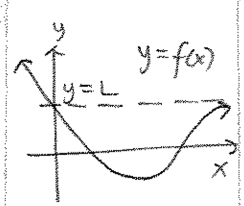
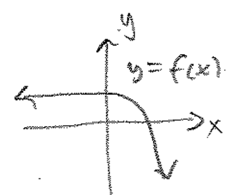
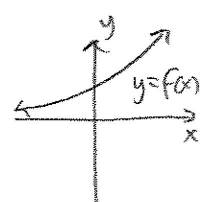
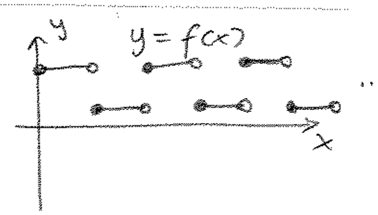
(g) $\lim_{x \to 0^+} \frac{\sin x}{1-\cos x} = \lim_{x \to 0^+} \frac{\sin x}{1-\cos x} \cdot \left(\frac{1+\cos x}{1+\cos x}\right)$

$= \lim_{x \to 0^+} \frac{(\sin x)(1+\cos x)}{1-\cos^2 x} = \lim_{x \to 0^+} \frac{\sin x\,(1+\cos x)}{\sin^2 x}$

$= \lim_{x \to 0^+} \frac{1+\cos x}{\sin x} = \infty$ (The numerator $\to 2$, the denominator $\to 0$)

$$\lim_{x \to A} f(x) = ?$$

7.

limit is / A =	L	$-\infty$	$+\infty$	fails to exist
a				
a^-				
a^+				
$-\infty$				
$+\infty$				

Many other possibilities exist!

18. $\lim_{v \to c^-} m(v) = \lim_{v \to c^-} \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = ?$

Notice that as $v \to c^-$, $\frac{v^2}{c^2} \to 1^-$, so $\left[1 - \left(\frac{v^2}{c^2}\right)\right] \to 0^+$.

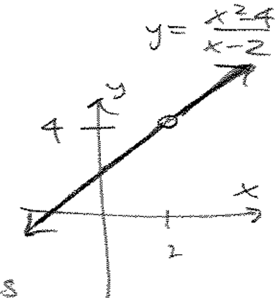
$\therefore \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \to +\infty$.

Continuity of Functions

19. (a) $f(x) = (x+1)(x-2)$ is continuous at 2 because f is a polynomial.

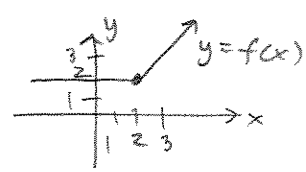
(b) $f(x) = \dfrac{x+1}{x-2}$ is not continuous at 2 because f is not defined at 2.

(c) $f(x) = \dfrac{x^2-4}{x-2}$ is not continuous at 2 because f is not defined at 2. *Note: this is a removable discontinuity — if we add $f(2) = 4$, then f would be continuous at 2.



(d) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$ is continuous (see the note above).

(e) $f(x) = \begin{cases} x & \text{if } x > 2 \\ 2 & \text{if } x \leq 2 \end{cases}$ is continuous



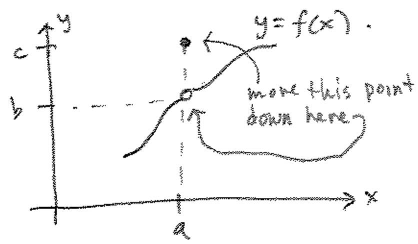
(f) $f(x) = [\![x]\!]$ is not continuous at 2, because $\lim_{x \to 2} f(x)$ is not defined.

20. f is continuous on $(-\infty, 3)$; $[-3, -1]$; $(-1, 1.5)$; $(1.5, \infty)$

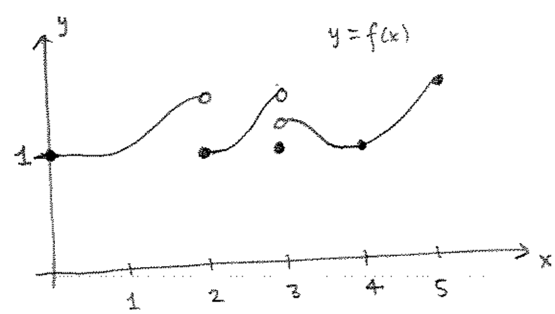
21.

Function	Point x where undefined	Required Value (the limit)
(a) $f(x) = \frac{x^2-9}{x+3}$	$x = -3$	$f(-3) = -6$
(b) $g(x) = \frac{1-\cos x}{x}$	$x = 0$	$g(0) = 0$
(c) $h(x) = \frac{\sin x}{x}$	$x = 0$	$h(0) = 1$
(d) $k(x) = x\sin(\frac{1}{x})$	$x = 0$	$k(0) = 0$
(e) $m(x) = \frac{x^2}{x(2+\cos x)}$	$x = 0$	$m(0) = 0$.

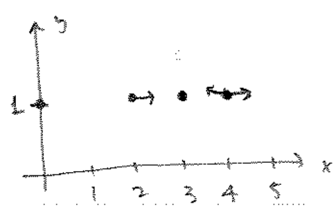
22. Change $f(a) = c$ to $f(a) = b = \lim_{x\to a} f(x)$.



23.



Many other possibilities.
It must have:



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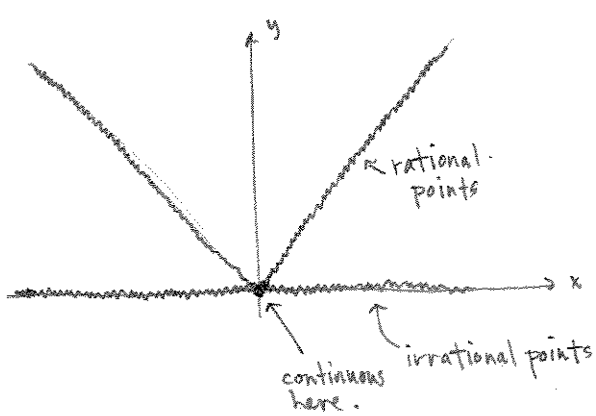


$0 \le f(x) \le |x|$ for all x.

and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} |x|$

So by the Squeeze Theorem,

$\lim_{x\to 0} f(x) = 0$.

But $f(0) = 0$, so f is continuous at 0.

25. sine is a continuous function, so we can pass a limit inside:

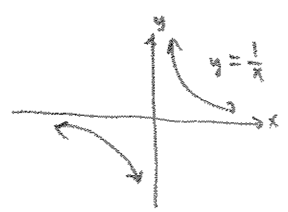
$$\lim_{x\to a} \sin(f(x)) = \sin\left(\lim_{x\to a} f(x)\right) = \sin b.$$

26. Let $p(x) = x^3 + 3x^2 - x - 3$. Then $p(0) = -3 < 0$

$$p(2) = 8 + 12 - 2 - 3$$
$$= 15 > 0.$$

p is continuous because it is a polynomial. So, by IVT, $\exists\, c \in [0,2]$ with $p(c) = 0$.

27. No contradiction. $f(x) = \frac{1}{x}$ is not continuous on $[-1,1]$



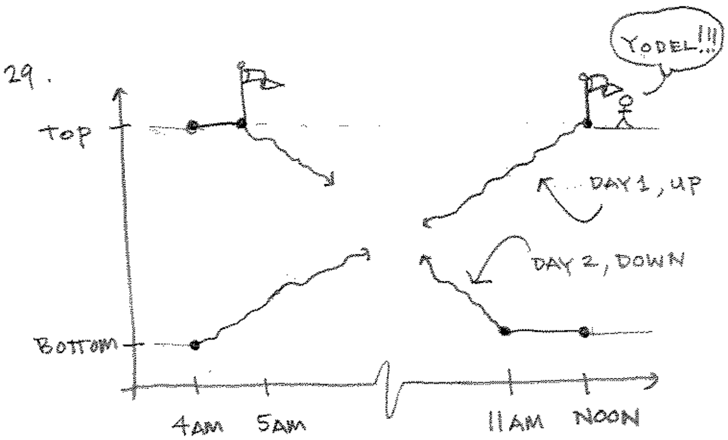
28. Let $g(x) = f(x) - x$. If we can find $c \in [0,1]$ s.t. $g(c) = 0$ then $f(c) = c$, and we will be done.

$f(0) \geq 0$. If $f(0) = 0$ we have $c = 0$ and we're done.
If $f(0) \neq 0$, then $f(0) > 0$.
so $g(0) = f(0) - 0 > 0$.

$f(1) \leq 1$. If $f(1) = 1$ we have $c = 1$ and we're done.
If $f(1) \neq 1$, then $f(1) < 1$
so $g(1) = f(1) - 1 < 0$

$g(x)$ is continuous because $f(x)$ and x are. Therefore, by IVT, $\exists\, c \in [0,1]$ with $g(c) = 0 \Rightarrow f(c) = c$. ∎

29.



We need to show that these two curves cross. It seems obvious, but we will use IVT to make it rigorous.

Let t be any time on our hiker's watch. We really are interested in $4\text{AM} \leq t \leq \text{NOON}$.

Let $f(t)$ = her position along the trail on DAY 2 at time t
minus
her position on DAY 1 at time t,

with the bottom of the trail being distance 0 units, and the top of the trail being distance 1 unit.

Just to be clear, here are some practice values:

$$f(4\text{AM}) = 1 - 0 = 1$$

$$f(\text{NOON}) = 0 - 1 = -1.$$

But these values are great! Now, because our hiker changes her position as a continuous function (she doesn't teleport), we can apply IVT: $f(t)$ is continuous, so there is a time

$$c \in [4\text{AM}, \text{NOON}]$$

for which $f(c) = 0$. So, at this time $t = c$, she is at the same point on the trail.

$\blacksquare$

The Derivative.

30. Let $s(t) = 5t^2$

(a) $s(0) = 0$, $s(1) = 5$. So $s(1) - s(0) = \boxed{5 \text{ meters.}}$

(b) $s(2) = 20$. So $s(2) - s(1) = \boxed{15 \text{ meters.}}$

(c) $s(4) = 80m$, $s(5) = 125m$.

$$v_{avg} = \frac{125m - 80m}{1 \sec} = \boxed{45 m/sec.}$$

(d) $s(4.01) = 80.4005$

$$v_{avg} = \frac{80.4005m - 80m}{0.01 \sec.} = \boxed{40.05 m/s}$$

(e)
$$v_{inst.} = \lim_{h\to 0} \frac{5(4+h)^2 - 5(4)^2}{h}$$

$$= \lim_{h\to 0} \frac{5(16+8h+h^2) - 5(16)}{h}$$

$$= \lim_{h\to 0} \frac{40h + 5h^2}{h} = \lim_{h\to 0} 40 + 5h = \boxed{40 m/s}.$$

31. (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \left(\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\right) = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) $f(x) = \frac{3}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \lim_{h\to 0} \frac{\frac{3x - 3(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}x(x+h)} = -\frac{3}{x^2}.$$

31 (c) $f(x) = \dfrac{x}{x+1}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x+1) - x(x+h+1)}{(x+1)(x+h+1)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{x^2 + (h+1)x + h - x^2 - xh - x}{(x+1)(x+h+1)}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \frac{\cancel{h}}{(x+1)(x+h+1)}$$

$$= \lim_{h \to 0} \frac{1}{(x+1)(x+h+1)} = \boxed{\frac{1}{(x+1)^2}} = \frac{1}{x^2+2x+1}$$

32. (a) $f'(0) = 0$

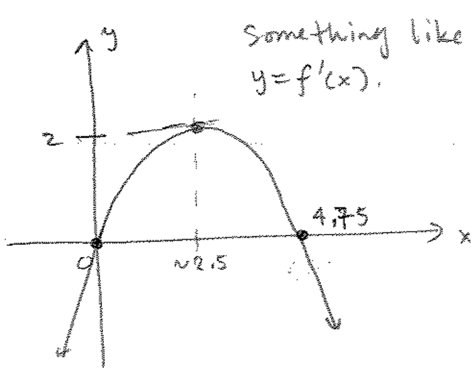
$f'(2) \simeq 2$

$f'(4) \simeq 1$

$f'(6) \simeq -\frac{3}{2}$

(b) $x = 0$ and $x \simeq 4.75$

$f'(x) = 0$ at these points.

(c) 

Something like this. $y = f'(x)$.

33.  $= f(x)$

 $= f'(x)$

34. $r_{ave.} = \dfrac{800\text{ kgal} - 100\text{ kgal}}{24\text{ hrs.}}$

$\cong \boxed{29\text{ kgal/hr}}$

(It is positive!)

$r_{8am} \simeq \dfrac{200\text{ kgal}}{4\text{ hrs.}}$

$= \boxed{50\text{ kgal/hr}}$

Rules for Finding Derivatives

35. Many examples possible.
must take functions to functions (or numbers)
and must be linear:

$$L(af+g) = a\,Lf + Lg \,.$$

36. (a) $D_x y = 6x$

(b) $D_x y = k$

(c) $D_x y = 9x^8 + 7x^6 + 5x^4 + 3x^2 + 1$

(d) $D_x y = -\dfrac{1}{x^2} + 2x$

(e) $D_x y = -3\pi x^{-4}$

(f)
$$\begin{aligned} D_x y &= (2x+1)(x-1) + (x^2+x+1)\,1 \\ &= 2x^2 - 2x + x - 1 + x^2 + x + 1 \\ &= 3x^2. \end{aligned}$$

Easy way:

$$y = (x^2+x+1)(x-1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1 \,.$$

$$D_x y = 3x^2.$$

(g)
$$\begin{aligned} D_x y &= \frac{1\,(x^2+x+1) - (2x+1)(x-1)}{(x-1)^2} \\ &= \frac{x^2+x+1 - 2x^2+2x-x+1}{(x-1)^2} \\ &= \frac{-x^2+2x+2}{(x-1)^2} \end{aligned}$$

(h)
$$y = 17\,\frac{x^4 + x^3}{x^2 - x + 1}$$

You can stop here.

$$D_x y = 17\,\frac{(4x^3+3x^2)(x^2-x+1) - (x^4+x^3)(2x-1)}{(x^2-x+1)^2}$$

$$= 17\,\frac{4x^5 - 4x^4 + 4x^3 + 3x^4 - 3x^3 + 3x^2 - 2x^5 + x^4 - 2x^4 + x^3}{(x^2-x+1)^2}$$

$$= 17\,\frac{2x^5 - 2x^4 + 2x^3 + 3x^2}{(x^2-x+1)^2} = \boxed{17x^2\,\frac{2x^3 - 2x^2 + 2x + 3}{(x^2-x+1)^2}}$$

36. (i) $\boxed{D_x y = -\frac{6}{23} x^{-3}}$

(j) $\boxed{D_x y = -10x^{-6} - 9x^{-4} - x^{-2}}$

(k) $D_x y = \boxed{987{,}654{,}321\, x^{987654320}}$

(l) $y = (x-4)^2 = x^2 - 8x + 16$

$D_x y = 2x - 8 = \boxed{2(x-4)}$

(m) $y = (x^2+1)^3 = (x^2)^3 + 3(x^2)^2 + 3(x^2)^1 + 1 = x^6 + 3x^4 + 3x^2 + 1$

Binomial Theorem

1
1 1
1 2 1
1 3 3 1

$\boxed{D_x y = 6x^5 + 12x^3 + 6x}$

$= 6x(x^4 + 2x^2 + 1)$

$= 6x(x^2+1)^2.$

37 (a) $D_x[f(x)]^2 = D_x[f(x) \cdot f(x)] =$

$= D_x f(x) \cdot f(x) + f(x) \cdot D_x f(x)$

$= 2f(x) D_x f(x)$ $\blacksquare$

(b) $D_x[f(x)]^3 = D_x[f(x) \cdot (f(x))^2]$

$= D_x f(x) \cdot (f(x))^2 + f(x) \cdot (2f(x) D_x f(x))$

$= 3(f(x))^2 D_x f(x).$ $\blacksquare$

(c) $D_x[f(x)]^n = n(f(x))^{n-1} D_x f(x).$

38. (a) $(f+g)'(1) = f'(1) + g'(1) = 2 + 4 = \boxed{6}$

(b) $(f \cdot g)'(1) \neq f'(1)g'(1)$!!!

$(f \cdot g)'(1) = f'(1)g(1) + f(1)g'(1) = 2 \cdot 3 + 1 \cdot 4 = \boxed{10}$

(c) $(f/g)'(1) = \dfrac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \dfrac{2 \cdot 3 - 1 \cdot 4}{3^2} = \boxed{\dfrac{2}{9}}$

39. Let $F(x) = f(x) - g(x)$.

$$F'(x) = \lim_{h \to 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) - (g(x+h) - g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x) \qquad \blacksquare$$

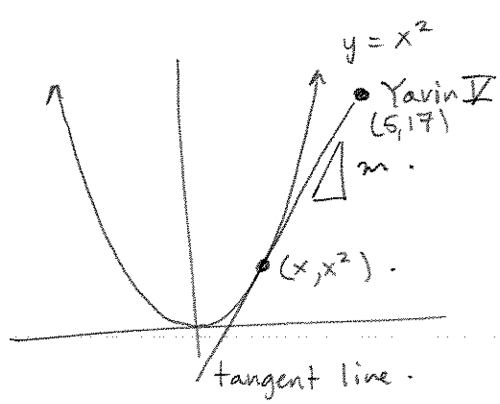
40. We want $y' = 0$.

$$y' = \tfrac{1}{3} \cdot 3x^2 + \tfrac{1}{2} \cdot 2x + 0$$

$$= x^2 + x.$$

$$x^2 + x = 0 \iff x(x+1) = 0 \iff \boxed{x = 0 \text{ or } x = -1}$$

41.



We need to compute the slope in two ways:

(i) $m = \frac{\text{rise}}{\text{run}} = \frac{17 - x^2}{5 - x}$

(ii) $m = y' = 2x$.

$$\Longrightarrow \quad 2x = m = \frac{17 - x^2}{5 - x} \iff 2x(5 - x) = 17 - x^2$$

$$\iff 10x - 2x^2 = 17 - x^2$$

$$x^2 - 10x + 17 = 0.$$

Quadratic Formula:

$$x = \frac{10 \pm \sqrt{100 - 4(17)}}{2}$$

$$= \frac{10 \pm \sqrt{32}}{2} = \frac{10 \pm 4\sqrt{2}}{2}$$

$$= 5 \pm \sqrt{2}.$$

If Chewie is traveling from left to right, he should cut power at $5 - \sqrt{2}$.
If going the other way, $5 + \sqrt{2}$

42. $y_1 = x^2$ $\qquad y_2 = -x^2 + \frac{3}{2}x + 1$

$m_1 = y_1' = 2x$ $\qquad m_2 = y_2' = -2x + \frac{3}{2}$.

For the tangent lines to be $\perp$,

we need

$$m_2 = -\frac{1}{m_1}.$$

$$\Longrightarrow \quad -2x + \frac{3}{2} = -\frac{1}{x}.$$

$$\Longrightarrow \quad x\left(2x - \frac{3}{2}\right) = 1$$

$$2x^2 - \frac{3}{2}x - 1 = 0$$

$$4x^2 - 3x - 2 = 0$$

Use Quadratic Formula:

$$x = \frac{3 \pm \sqrt{9 - 4(4)(-2)}}{2(4)}$$

$$= \frac{3 \pm \sqrt{9 + 32}}{8}$$

$$= \boxed{\frac{3 \pm \sqrt{41}}{8}}.$$

Derivatives of Trig Functions

43. (a) $D_x y = \boxed{3\cos x - 2\sin x}$

(b) $D_x y = D_x(\sin x \cdot \sin x \cdot \sin x)$

$= \cos x \cdot \sin^2 x + \cos x \cdot \sin^2 x + \cos x \cdot \sin^2 x$

$= \boxed{3\sin^2 x \cos x}$.

(c) $D_x y = \boxed{\sec^2 x + \cos x}$.

(d) $D_x y = (\cos x)(\cos x) + (\sin x)(-\sin x)$

$= \boxed{\cos^2 x - \sin^2 x}$.

(e) $$D_x y = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\cos x + \sin x)(\sin x + \cos x)}{(\sin x - \cos x)^2}$$

any of these

$$\boxed{\begin{aligned} &= -1 - \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^2 \\ &= -1 - \left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)^2 \end{aligned}}$$

Note that this is $D_x y = -1 - y^2$.

(f) $\boxed{D_x y = 1 \cdot \sin x + x\cos x}$

(g) $\boxed{D_x y = 3x^2 \cos x - x^3 \sin x}$

(h) Let $f(x) = x\sin x + \cos x$, $g(x) = x^2 + 1$

$f'(x) = \sin x + x\cos x - \sin x$ $\quad g'(x) = 2x$.

$= x\cos x$

Then $$D_x y = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{x\cos x\,(x^2+1) - 2x(x\sin x + \cos x)}{(x^2+1)^2} = \boxed{\frac{x^3\cos x - x\cos x - 2x^2\sin x}{(x^2+1)^2}}$$

(i) $y = 1$. So $\boxed{D_x y = 0}$ $\quad$ (j) $\boxed{D_x y = -\sin x - 1}$

44. $f(x) = \cos(3x)$

Use
$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

$$f'(x) = \lim_{h\to 0} \frac{\cos(3x+3h) - \cos(3x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(3x)\cos(3h) - \sin(3x)\sin(3h) - \cos(3x)}{h}.$$

$$= \lim_{h\to 0} -\cos(3x)\frac{(1-\cos(3h))}{h} - \sin(3x)\frac{\sin(3h)}{h}$$

$$= \lim_{h\to 0} -3\cos(3x)\frac{(1-\cos(3h))}{3h} - 3\sin(3x)\frac{\sin(3h)}{3h}$$

Let $k = 3h$
$k \to 0$ as $h \to 0$.

$$= \lim_{k\to 0} -3\cos(3x)\frac{1-\cos k}{k} - 3\sin 3x\frac{\sin k}{k}$$

$$= -3\cos(3x)\cdot 0 - 3\sin(3x)\cdot 1$$

$$= \boxed{-3\sin 3x}$$

45. $y = \cos x$.

$y(\pi/2) = \cos(\pi/2) = 0$. So $(\pi/2, 0)$ is on our line.

$y' = -\sin x$.

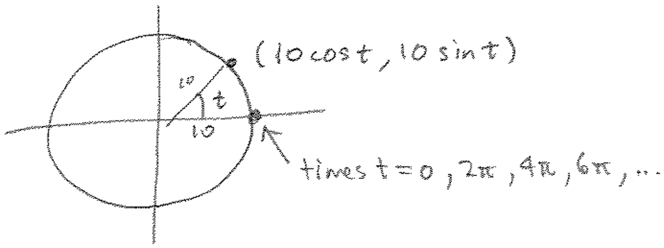
$y'(\pi/2) = -\sin(\pi/2) = -1$.

So our slope $m = 1$.

Using Point-Slope Form: $y - y_1 = m(x - x_1)$

$$y - 0 = 1(x - \pi/2)$$

$$\boxed{y = x - \pi/2}$$

46. 

(a) $x = 10\cos t$ meters

$y = 10\sin t$ meters

(b) $\frac{dy}{dt} = 10\cos t$ m/s

(c) It is rising most quickly at

$t = 0$ sec, 2π sec, 4π sec, ...

Its vertical speed at these times is 10 m/s.

The Chain Rule

47. (a) $f'(x) = 99(x+1)^{98} \cdot 1$

(b) $f'(x) = 2x\cos(x^2)$

(c) $\frac{dw}{dt} = 100(t^2+1)^{99} \cdot 2t$

$= 200t(t^2+1)^{99}$

(d) $\frac{dw}{dt} = 300t^2(t^2+1)^{99}$

(e) $f'(x) = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$= \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$

(f) $g'(t) = \sec^2(\cos t) \cdot (-\sin t)$

$= -\sin t \sec^2(\cos t)$.

(g) $f'(x) = 2\sin x \cos x$.

(h) $h'(t) = -2\sin^{-3}(t) \cdot \cos t$

$= -2\frac{\cos t}{\sin^3 t}$

(i) $\frac{dy}{dx} = 3\sin^2(x^2+x+1)(2x+1)$

(j) $\frac{dz}{dx} = n\cos^{n-1}x\,(-\sin x)$

$= -n\sin x\cos^{n-1}x$

(k) $\frac{dw}{dx} = 2x\sin(x^2) + x^2\cos(x^2) \cdot 2x$

$= 2x\sin(x^2) + 2x^3\cos(x^2)$.

48. (a) $h(4) = f(g(4)) = f(2) = 1$

(b) $h'(4) = f'(g(4)) \cdot g'(4)$

$= f'(2) \cdot 6$

$= 5 \cdot 6 = 30$

(c) $h(4) = g(f(4)) = g(3) = 4$

(d) $h'(4) = g'(f(4)) \cdot f'(4)$

$= g'(3) \cdot 7$

$= 8 \cdot 7 = 56.$

(e) $h'(4) = \dfrac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2}$

$= \dfrac{7 \cdot 2 - 3 \cdot 6}{2^2}$

$= \dfrac{14 - 18}{4}$

$= -2.$

49.

$r(t) = 10t$ cm

$A = \pi r^2 = 100\pi t^2.$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi(20\,\text{cm}) \cdot \left(\frac{10\,\text{cm}}{\text{sec}}\right)$$

$$= 400\pi\ \text{cm}^2/\text{sec}.$$

50. Let $f(x)$ be odd.

Then $f(-x) = -f(x)$.

Apply $\frac{d}{dx}$: $-f'(-x) = -f'(x)$

so $f'(-x) = f'(x)$.

Thus $f'(x)$ is even.

Let $g(x)$ be even.

Then $g(-x) = g(x)$

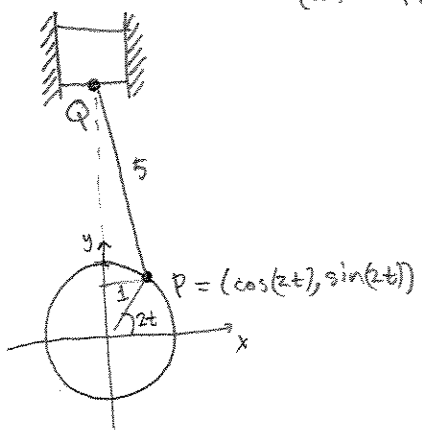
Apply $\frac{d}{dx}$: $-g'(-x) = g'(x)$

So $g'(-x) = -g'(x)$

Thus $g'(x)$ is odd.

∎

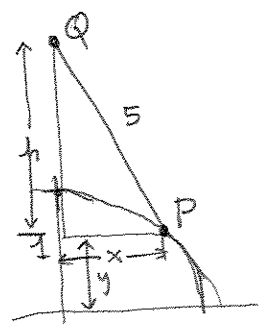
51.



(a) $P(t)$:

$$x_p(t) = \cos(2t)$$
$$y_p(t) = \sin(2t)$$

(b)



The y-coordinate of Q is h plus (1 minus the y-coordinate of P).

So $y_Q(t) = h(t) + 1 - \sin(2t)$.

But $(h(t))^2 + (x_p(t))^2 = (5)^2$

so $h(t) = \sqrt{25 - \cos^2(2t)}$

$$y_Q(t) = \sqrt{25 - \cos^2(2t)} + 1 - \sin(2t)$$

(c) $$v_Q(t) = \frac{dy_Q}{dt} = \frac{1}{2\sqrt{25 - \cos^2(2t)}} \cdot \left(-2\cos(2t)(-\sin(2t))(2)\right) + 0 - \cos(2t) \cdot 2$$

$$= \boxed{\frac{2\cos(2t)\sin(2t)}{\sqrt{25-\cos^2(2t)}} - 2\cos(2t)}$$

$$= \frac{2x_p y_p}{\sqrt{25 - x_p^2}} - 2x_p .$$

Leibniz Notation

52. (a) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2(2x+1)$
$= 3(x^2+x+1)^2(2x+1)$

(b) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot (-\sin x)$
$= -2\cos x \sin x$

(c) $\frac{dy}{dx} = \cos\left(\frac{1}{x+1}\right) \cdot \frac{-1}{(x+1)^2} = -\frac{\cos\left(\frac{1}{x+1}\right)}{(x+1)^2}$

(d) $\frac{dy}{dx} = 57\sin^{56}(x^3+2x^2+1)\cos(x^3+2x^2+1)(3x^2+4x)$

(e) $\frac{dy}{dx} = 2\tan(\cos x)\sec^2(\cos x)(-\sin x)$
$= -2\tan(\cos x)\sec^2(\cos x)(\sin x)$.

(f) $\frac{dy}{dx} = -\sin\left((x^2+1)^4\right) \cdot 4(x^2+1)^3 \cdot (2x)$
$= -8x(x^2+1)^3\sin\left((x^2+1)^4\right)$.

53.

x	u	$\frac{du}{dx}$	y	$\frac{dy}{du}$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
0	0	4	3.5	1	4
1	3	2	0.5	$\frac{2}{3}$ or $\frac{3}{4}$	about 1.33 to 1.5
2	4	0	3.5	finite	0
3	3	−2	0.5	$\frac{2}{3}$ or $\frac{3}{4}$	about −1.5 to −1.33
4	0	−4	3.5	1	−4.

These are all approximations!

54. $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$, $\frac{dr}{dt} = 40{,}000$ km/hr

$r = 1{,}000{,}000$ km

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi\,(1\times10^6 \text{ km})^2 \cdot (4\times10^4 \text{ km/hr})$$

$$= 16\pi \times 10^{16} \text{ km}^3/\text{hr}$$ (Note: $16\pi \simeq 50$)

$$\simeq 500{,}000{,}000{,}000{,}000{,}000 \text{ km}^3/\text{hr}.$$

55.
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) If $x \neq 0$, then $f'(x) = 2x\sin\left(\frac{1}{x}\right) - x^2\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)$

$$= 2x\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right).$$

(b) If $x = 0$, we must use the definition.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

by the Squeeze theorem! (see Problem 13)

(c) So
$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

But
$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right),$$ which D.N.E.

56.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot(g(x))^{-1}\right)$$

$$= f'(x)\cdot(g(x))^{-1} + f(x)\cdot\left(-1(g(x))^{-2}\right)\cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{(g(x))^2} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}.$$

Higher Order Derivatives

57. (a) $\frac{dy}{dx} = 3x^2 + 16x^3 + 3$

$$\frac{d^2y}{dx^2} = 6x + 48x^2$$

$$\frac{d^3y}{dx^3} = 6 + 96x .$$

(b) $\frac{d^3y}{dx^3} = 0$

(c) $\frac{d^3y}{dx^3} = 27\sin(3x)$

(d) $\frac{dy}{dx} = 2x\cos(x^2)$

$$\frac{d^2y}{dx^2} = 2\cos(x^2) - 4x^2\sin(x^2)$$

$$\frac{d^3y}{dy^3} = -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2)$$

$$= -12x\sin(x^2) - 8x^3\cos(x^2)$$

$$= -4x\left(3\sin(x^2) + 2x^2\cos(x^2)\right) .$$

(e) $\frac{dy}{dx} = -\frac{2x}{(x^2+1)^2}$

$$\frac{d^2y}{dx^2} = -2(x^2+1)^{-2} + 4x(x^2+1)^{-3} \cdot 2x$$

$$= 2(x^2+1)^{-3}\left(4x^2 - (x^2+1)\right)$$

$$= 2(x^2+1)^{-3}\left(3x^2 - 1\right)$$

58. (a) $v(t) = s'(t) = 9t^2 - 10t + 1$

$a(t) = s''(t) = v'(t) = 18t - 10$

(b) when $v(t) > 0$.

$v(t) = 9t^2 - 10t + 1 = (1-t)(1+9t)$

$v(t) > 0$ when $-\frac{1}{9} < t < 1$ $\quad t \in (-\frac{1}{9}, 1)$

(c) when $v(t) < 0$

$v(t) < 0$ when $t \in (-\infty, -\frac{1}{9}) \cup (1, \infty)$

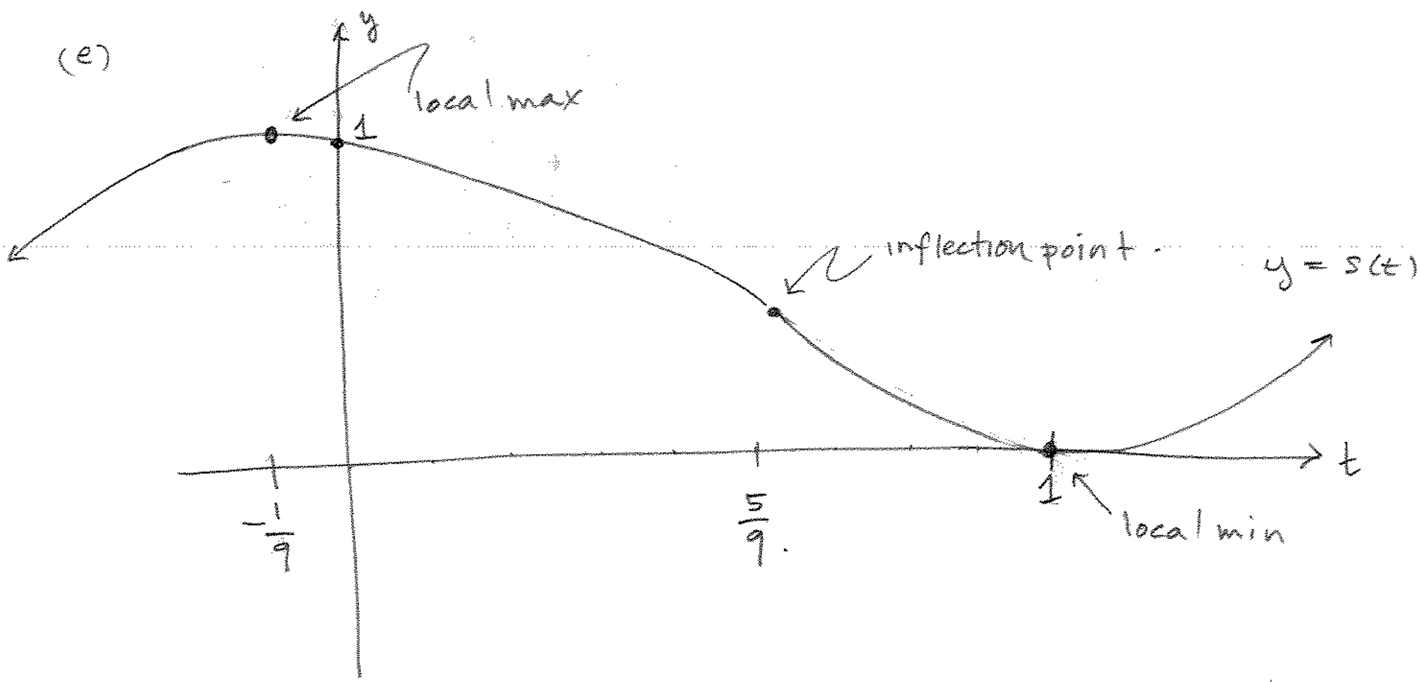
or $\quad t < -\frac{1}{9}$ or $t > 1$.

(d) when $a(t) < 0$.

$a(t) < 0$ when $18t < 10$

when $t < \frac{5}{9}$.

(e)



59. $s = 20t - 5t^2$

(a) $v(t) = s'(t) = 20 - 10t$

$v(0) = 20$ m/sec.

(b) when $v(t) = 0$.

$\Rightarrow$ $t = 2$ sec.

(c) $s(2) = 40 - 20 = 20$ m.

(d) $v(1) = 15$ m/sec. upward.

(e) when $s(t) = 0$.

$s(t) = 5t(4 - t)$.

when $t = 0$ $\leftarrow$ launch

$t = 4$ $\leftarrow$ landing.

(f) $v(4) = 20 - 10(4) = -20$ m/sec.

which is 20 m/sec in the downward direction.

Implicit Differentiation

60. (a) $x^2+y^2=1$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

(b) $xy=1$

$$y+x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

(c) $x^2+y^2=x+1$

$$2x+2y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1-2x}{2y}$$

(d) $\cos(xy)+\sin(xy)=\sqrt{2}$

$$\Big(-\sin(xy)+\cos(xy)\Big)\left(y+x\frac{dy}{dx}\right)=0.$$

$$\frac{dy}{dx}=-\frac{y}{x} \quad \text{provided} \quad \cos(xy)\neq\sin(xy).$$

(e) $x^2y+y^2x=0$

$$2xy+x^2\frac{dy}{dx}+2y\frac{dy}{dx}x+y^2=0$$

$$\frac{dy}{dx}\left(x^2+2xy\right)=-\left(y^2+2xy\right)$$

$$\frac{dy}{dx}=-\frac{y^2+2xy}{x^2+2xy}.$$

61. (a) $\left.\frac{dy}{dx}\right|_{\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)}=1$

(b) $\left.\frac{dy}{dx}\right|_{\left(\frac{1}{4},4\right)}=-16$

(c) $\left.\frac{dy}{dx}\right|_{(1,\sqrt{2})}=-\frac{1}{2\sqrt{2}}$

(d) $\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{4},1\right)}=-\frac{4}{\pi}$

Actually, it is undefined, but its limit is $-4/\pi$.

(e) $\left.\frac{dy}{dx}\right|_{(-1,1)}=-1$

62. (a) $y = x^{1/2} - x^{9/5}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{9}{5}x^{4/5}$$

(b) $y = x^{-2/3}$

$$\frac{dy}{dx} = -\frac{2}{3}x^{-5/3}$$

(c) $y = (x^2+1)^{-1/2}$

$$\frac{dy}{dx} = -x(x^2+1)^{-3/2}$$

63. $x^2+y^2 = 25$. Apply $\frac{d}{dx}$.

$$2x + 2yy' = 0 \implies y' = -\frac{x}{y}.$$

$$\Downarrow$$

$$x + yy' = 0$$

Apply $\frac{d}{dx}$ again:

$$1 + y'\cdot y' + yy'' = 0$$

$$y'' = -\frac{1+(y')^2}{y} = -\frac{1+\frac{x^2}{y^2}}{y}$$

$$= -\frac{x^2+y^2}{y^3}.$$

At $(4,3)$, $y'' = -\frac{5}{27}$.

64. $s^3t + s^2t^2 = 1$

Apply $\frac{d}{dt}$.

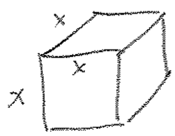
$$3s^2\frac{ds}{dt}\cdot t + s^3 + 2s\frac{ds}{dt}t^2 + s^2 2t = 0.$$

$$\frac{ds}{dt} = -\frac{s^3+2s^2t}{3s^2t+2st^2}.$$

Applying $\frac{d}{ds}$ yields

$$\frac{dt}{ds} = -\frac{3s^2t+2st^2}{s^3+2s^2t},$$

Related Rates

65. 

$V = x^3 \qquad \frac{dx}{dt} = 5\text{ cm/sec}.$

$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} \qquad @\ x = 20\text{ cm}.$

$= 3x^2 \cdot 5\text{ cm/sec}.$

$= 3(20\text{ cm})^2 \cdot 5\text{ cm/sec}.$

$= \boxed{6000\text{ cm}^3/\text{sec}.}$

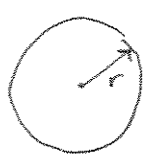
$A = 6x^2.$

$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} \qquad @\ x = 20\text{ cm}.$

$= 12x \cdot 5\text{ cm/sec}.$

$= 12(20\text{ cm})(5\text{ cm/sec}).$

$= \boxed{1200\text{ cm}^2/\text{sec}}$

66. 

$\frac{dr}{dt} = 1\text{ mm/sec}.$

$A = \pi r^2$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} \qquad @\ r = 100\text{ mm}.$

$= 2\pi r \cdot \frac{dr}{dt}$

$= 2\pi(100\text{ mm})(1\text{ mm/sec}).$

$\left.\begin{array}{l} = 200\pi\text{ mm}^2/\text{sec}. \\ = 2\pi\text{ cm}^2/\text{sec}. \end{array}\right\}\text{either}$

67.  $\frac{dV}{dt} = 12\ ft^3/sec.$ $V = \frac{1}{3}\pi r^2 h.$ $h = \frac{1}{4}d = \frac{1}{2}r$

$\Longrightarrow r = 2h.$

$\Longrightarrow V = \frac{4}{3}\pi h^3$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}.$$

So $$\frac{dh}{dt} = \left(\frac{dV}{dt}\right) \Big/ \left(\frac{dV}{dh}\right) = \frac{(12\ ft^3/sec.)}{(4\pi h^2)}$$

$$= \frac{12\ ft^3/sec}{64\pi\ ft^2}$$

$$= \frac{3}{16\pi}\ ft/sec.$$

$$\approx 0.6\ ft/sec.$$

68. Skip!

69. $V = \frac{4}{3}\pi r^3$ $R(t) = 10 + 40000t - 10t^2$

(a) $R(0) = 10$ km.

(b) $R'(t) = 40000 - 20t$

$R'(0) = 40000$ km/hr.

(c) $V(0) = \frac{4}{3}\pi(R(0))^3 = \frac{4000\pi}{3}\ km^3$

(d) $\frac{dV}{dt} = \frac{dV}{dR} \cdot \frac{dR}{dt} = 4\pi R^2 \cdot \frac{dR}{dt}$

$\left.\frac{dV}{dt}\right|_{t=0} = (400\pi\ km^2)(40000\ km/hr)$

$= 16{,}000{,}000\pi\ km^3/hr.$

(e) $R'(1) = 39{,}980$ km/hr. Note: $R(1) = 40000$ km.

(f) $\left.\frac{dV}{dt}\right|_{t=1} = 4\pi(40000\ km)^2 \cdot (39{,}980\ km/hr) =$ __________

(g) $R'(24) = 40000 - 480 = 39{,}520$ km/hr. Note: $R(24) =$ __________

(h) $\left.\frac{dV}{dt}\right|_{t=24} =$

(i) $V(24) = \frac{4}{3}\pi(R(24))^3 =$