

Math 1210 Homework

Summer 2008

Note that the symbol \boxplus indicates that graph paper might be useful for that problem.

Review

Do the following problems, and check your answers in the back of the book. It is more important that you do problems that you do not know how to do than it is to do those you already understand very well. These problems won't be collected or graded. I will answer questions on these review problems in the problem session on Friday.

0.1 Real Numbers

5, 7, 9, 11, 13, 19, 21, 25, 27, 49, 51, 55.

0.2 Inequalities and Absolute Value

1, 3, 5, 11, 15, 23, 35, 37, 59.

0.3 Rectangular Coordinates

1, 3, 5, 7, 11, 13, 19, 35, 37, 39.

0.4 Graphs of Equations

1, 3, 5, 7, 9, 11, 13, 15, 17, 39.

0.5 Functions and Their Graphs

1, 3, 7, 8, 11, 13, 15, 35, 41, 42, 43, 44.

0.6 Operations on Functions

1, 11, 13, 15, 17, 23, 29.

0.7 Trigonometric Functions

1, 9, 15, 17, 19, 21, 25, 32, 47.

Introduction to Limits

1. Find the indicated limits.

(a) $\lim_{x \rightarrow 2} (x - 3)$

(b) $\lim_{x \rightarrow -1} (x^2 + 3x - 4)$

(c) $\lim_{x \rightarrow -3} \frac{x^2 - 4x - 21}{x + 3}$

(d) $\lim_{y \rightarrow 0} \frac{y^5 + 2y^4 + y^3}{y^3}$

(e) $\lim_{t \rightarrow -5} \frac{\sqrt{(t+5)^3}}{t+5}$

2. Below are two important limits. Use a calculator to estimate the limit by taking smaller and smaller values for x . Be sure to check both sides of 0.

(a) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

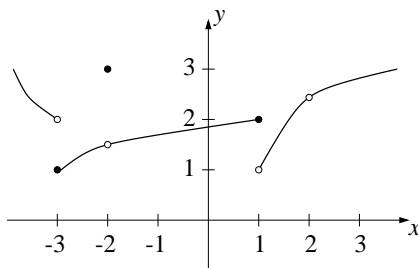
(b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

3. \boxplus Plot each of the following graphs on a small window near the origin. On each also plot $y = x$. Use these graphs to explain the results from Problem 2.

(a) $y = \sin x$

(b) $y = 1 - \cos x$

4. For the function f graphed in the figure below, find the indicated limit or function value, or state that it does not exist.



(a) $\lim_{x \rightarrow -2} f(x)$

(b) $f(-2)$

(c) $\lim_{x \rightarrow -3} f(x)$

(d) $f(-3)$

(e) $\lim_{x \rightarrow 1} f(x)$

(f) $f(1)$

(g) $\lim_{x \rightarrow 2} f(x)$

(h) $f(2)$

(i) $\lim_{x \rightarrow 1^+} f(x)$

(j) $\lim_{x \rightarrow -2^-} f(x)$

5. \boxplus Sketch, as best you can, the graph of a function f that satisfies all of the following properties:

(a) Its domain is the interval $[0, 4]$.

(b) $f(0) = f(1) = f(2) = f(3) = f(4) = 1$.

(c) $\lim_{x \rightarrow 1} f(x) = 2$.

(d) $\lim_{x \rightarrow 2} f(x) = 1$.

(e) $\lim_{x \rightarrow 3^-} f(x) = 2$.

(f) $\lim_{x \rightarrow 3^+} f(x) = 1$.

6. Suppose that you carefully graphed the function $f(x) = x^3$. Then, during the night, a prankster mathematician changed the values of f at a million different places on your graph. Did this affect the value of $\lim_{x \rightarrow a} f(x)$ at any a ? Explain.

Rigorous Study of Limits

7. \boxplus For each of the following limits, sketch a possible graph for the function f and give the appropriate ϵ - δ definition.
- (a) $\lim_{x \rightarrow 1} f(x) = 3$.
 - (b) $\lim_{x \rightarrow a} f(x) = L$.
 - (c) $\lim_{x \rightarrow 2^+} f(x) = 1$, $\lim_{x \rightarrow 2^-} f(x) = 3$.
 - (d) $\lim_{x \rightarrow 0^-} f(x) = 0$, $\lim_{x \rightarrow 0^+} f(x)$ does not exist.
 - (e) $\lim_{x \rightarrow c^+} f(x) = M$.
8. \boxplus Let f and g be functions such that $0 \leq f(x) \leq g(x)$ for all x near c , but possibly not at c . Prove that if $\lim_{x \rightarrow c} g(x) = 0$ then $\lim_{x \rightarrow c} f(x) = 0$. Draw a picture. This is a special case of the Squeeze Theorem from the next section, but you should prove this without using that theorem.

Limit Theorems

9. Evaluate each of the following limits using the limit theorems. State which theorem you use at each stage.

(a) $\lim_{x \rightarrow 1} (3x + 2)$.

(b) $\lim_{x \rightarrow 0} (2x + 5)(x + 1)$.

(c) $\lim_{x \rightarrow 2} \frac{x + 1}{x - 1}$.

(d) $\lim_{x \rightarrow -1} \sqrt{x^2 + 3}$.

(e) $\lim_{t \rightarrow 2} \left(\frac{7t + 3}{t^2 + t + 1} \right)^{1/5}$.

10. Suppose f and g are positive functions and $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = 2$. Find the following limits.

(a) $\lim_{x \rightarrow a} f(x) - g(x)$.

(b) $\lim_{x \rightarrow a} f^2(x) + g^2(x)$.

(c) $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{f(x) + g(x)}$.

(d) $\lim_{x \rightarrow a} (g^2(x) - f^2(x))^4$.

11. Find two functions f and g so that $\lim_{x \rightarrow 0} [f(x) + g(x)]$ exists but at least one of $\lim_{x \rightarrow 0} f(x)$ or $\lim_{x \rightarrow 0} g(x)$ does not exist.

12. Find two functions f and g so that $\lim_{x \rightarrow 0} [f(x) \cdot g(x)]$ exists but at least one of $\lim_{x \rightarrow 0} f(x)$ or $\lim_{x \rightarrow 0} g(x)$ does not exist.

13. \boxplus Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$. Draw a picture. Hint: Consider $f(x) = |x|$ and $g(x) = -|x|$.

14. \boxplus Show that $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.

Limits Involving Trigonometric Functions

15. Evaluate each of the following limits.

(a) $\lim_{t \rightarrow 0} \frac{\sin t}{1+t}$.

(b) $\lim_{t \rightarrow 0} t \sin t$.

(c) $\lim_{t \rightarrow 0} \frac{\sin^2 t}{t}$ (Hint: Use the Main Limit Theorem from last section together with a theorem from this section).

(d) $\lim_{t \rightarrow 0} \frac{1 - \cos^2 t}{t}$.

(e) $\lim_{t \rightarrow 0} \frac{\sin^2 t}{t(1 + \cos t)}$ (Hint: do some algebraic manipulation first).

Asymptotic Limits and Infinite Limits

16. Find the limits:

(a) $\lim_{x \rightarrow \infty} \frac{x}{x+2}$

(b) $\lim_{x \rightarrow \infty} \frac{x^2}{x^3+2}$

(c) $\lim_{x \rightarrow \infty} \frac{x^3}{x^2+2x+1}$

(d) $\lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta^2+1}$

(e) $\lim_{x \rightarrow -2^+} \frac{x}{x+2}$

(f) $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$

(g) $\lim_{x \rightarrow 0^+} \frac{\sin x}{1 - \cos x}$

17. We have given meaning to $\lim_{x \rightarrow A} f(x)$ for $A = a, a^-, a^+, -\infty, +\infty$. Moreover, this limit may be L (a finite number), $-\infty, +\infty$, or may fail to exist in any sense. This means that there are twenty possibilities. Give examples of $f(x)$ for each of these possibilities (either by formula, or by graph).

18. Einstein's Special Theory of Relativity says that the mass $m(v)$ of an object is related to its velocity v by

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where c is the speed of light in a vacuum, and m_0 is the rest mass of the object. What is $\lim_{v \rightarrow c^-} m(v)$?

Continuity of Functions

19. For each of the following functions, determine whether or not it is continuous at 2. If it is not continuous at 2, explain why not.

(a) $f(x) = (x + 1)(x - 2)$

(b) $f(x) = \frac{x + 1}{x - 2}$

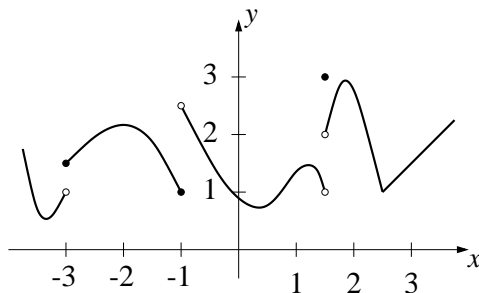
(c) $f(x) = \frac{x^2 - 4}{x - 2}$

(d) $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$

(e) $f(x) = \begin{cases} x & \text{if } x > 2 \\ 2 & \text{if } x \leq 2 \end{cases}$

(f) $f(x) = \lceil x \rceil$ (This is the “greatest integer” function).

20. From the graph $y = f(x)$ below, state the intervals on which $f(x)$ is continuous.



21. Each of the following functions are not defined somewhere. At what point are they not defined. We can explicitly define them at this point. What value should we assign so that the function is continuous at this point.

(a) $f(x) = \frac{x^2 - 9}{x + 3}$.

(b) $g(x) = \frac{1 - \cos x}{x}$.

(c) $h(x) = \frac{\sin x}{x}$.

(d) $k(x) = x \sin\left(\frac{1}{x}\right)$.

(e) $m(x) = \frac{x^3}{x(2 + \cos x)}$.

22. \boxplus Suppose that f is a function which is continuous everywhere except at a (where $f(x) = c$). Furthermore, $\lim_{x \rightarrow a} f(x) = b$ (and $b \neq c$). How can we get a function that agrees with f everywhere except at a , and is continuous everywhere? Draw a picture.

23. \boxplus Sketch the graph of a function f which satisfies all of the following properties:

(i) The domain of f is $[0, 5]$.

(ii) $f(0) = f(2) = f(3) = f(4) = 1$.

(iii) f is discontinuous at 2, 3, and 4.

(iv) f is right-continuous at 2, left-continuous at 4, and neither right- nor left-continuous at 3.

24. \boxplus Let

$$f(x) = \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Sketch a graph of f as best as you can and find the single point where f is continuous. Why is f continuous at this point?

25. Suppose that f is a function such that $\lim_{x \rightarrow a} f(x) = b$. Prove that $\lim_{x \rightarrow a} \sin(f(x)) = \sin b$.

26. Use the Intermediate Value Theorem to prove that $x^3 + 3x^2 - x - 3 = 0$ has a real solution between 0 and 2.

27. Let $f(x) = \frac{1}{x}$. Then $f(-1) = -1$ and $f(1) = 1$. Does the Intermediate Value Theorem imply that there is a point c in the interval $[-1, 1]$ such that $f(c) = 0$? Why or why not?

28. \boxplus Suppose that f is continuous on $[0, 1]$ and $0 \leq f(x) \leq 1$. Prove that f has a **fixed point** in $[0, 1]$. In other words, there exists c in $[0, 1]$ so that $f(c) = c$. Hint: Use the Intermediate Value Theorem and consider the function $g(x) = f(x) - x$.

29. Starting at 4 A.M., a hiker slowly climbed to Lone Peak Cirque, arriving at noon. The next day, she returned along the same path, starting at 5 A.M. and getting to the bottom at 11 A.M. Show that at some point along the path her watch showed the same time on both days.

The Derivative

30. A physics student determines through experiment that if she drops a ball, it will fall approximately $5t^2$ meters in t seconds after being released.

- How far will it fall between $t = 0$ and $t = 1$?
- How far will it fall between $t = 1$ and $t = 2$?
- What is the average velocity of the ball between the times $t = 4$ and $t = 5$?
- What is the average velocity of the ball between the times $t = 4$ and $t = 4.01$?
- What is the instantaneous velocity of the ball at time $t = 4$?

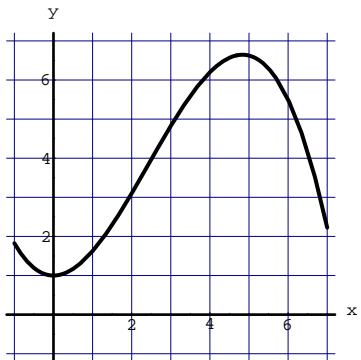
31. Use the definition

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

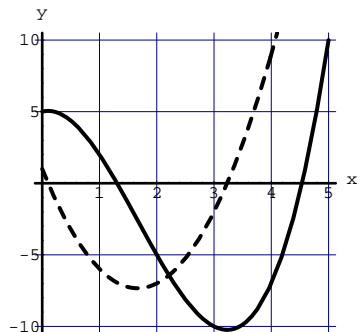
to find the derivative at x of each of the following functions:

- $f(x) = \sqrt{x}$
- $f(x) = \frac{3}{x}$
- $f(x) = \frac{x}{x+1}$

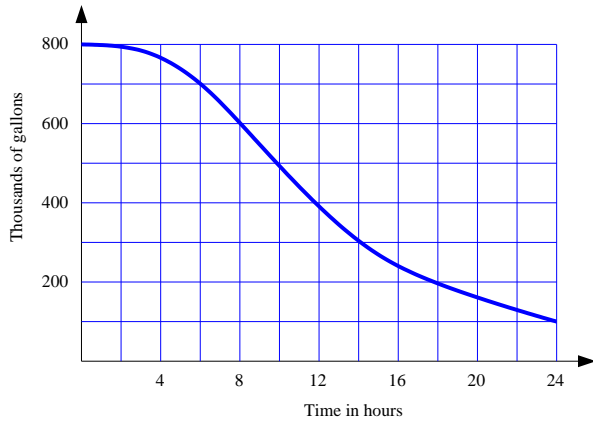
32. Consider graph of $y = f(x)$ in the following figure.



- Estimate $f'(0)$, $f'(2)$, $f'(4)$, $f'(6)$
 - There are two “special” points on this graph. For now let’s call them “vertices”. At one of the vertices, the graph is locally a minimum. This means that of all of the nearby points, it has the lowest value. Similarly, at the other vertex, we have a local maximum. What are the x -coordinates of these two points? What do you notice about the derivative at these two points?
 - Sketch the graph of $y = f'(x)$ on $-1 \leq x \leq 7$.
33. The Figure below shows the graphs of two functions. One is the function f and the other is its derivative f' . Which is which?



34. The graph below shows the amount of water in a city water tank during one day when no water was pumped into the tank. What was the average rate of water usage during the day? How fast was water being used at 8 A.M.?



Rules for Finding Derivatives

35. We learned that the derivative is a *linear operator*. Find another simple linear operator L and show that L is a linear operator. Hint: Remember that operators take functions as their inputs and output functions as well. Keep it simple.

36. Find $D_x y$ using the rules we have learned.

(a) $y = 3x^2$

(b) $y = kx$

(c) $y = x^9 + x^7 + x^5 + x^3 + x + 1$

(d) $y = \frac{1}{x} + x^2$

(e) $y = \frac{\pi}{x^3}$

(f) $y = (x^2 + x + 1)(x - 1)$

(g) $y = \frac{x^2 + x + 1}{x - 1}$

(h) $y = 17x^3 \left(\frac{x + 1}{x^2 - x + 1} \right)$

(i) $y = \frac{3}{23x^2} - \frac{13}{2}$

(j) $y = 2x^{-5} + 3x^{-3} + x^{-1}$

(k) $y = x^{987654321}$

(l) $y = (x - 4)^2$

(m) $y = (x^2 + 1)^3$

37. Use the Product Rule to show each of the following:

(a) $D_x [f(x)]^2 = 2 \cdot f(x) \cdot D_x f(x)$.

(b) $D_x [f(x)]^3 = 3 \cdot [f(x)]^2 \cdot D_x f(x)$.

(c) What do you think $D_x [f(x)]^n$ (where n is a positive integer) will be?

38. Suppose $f(1) = 1$, $f'(1) = 2$, $g(1) = 3$, $g'(1) = 4$. Find

(a) $(f + g)'(1)$

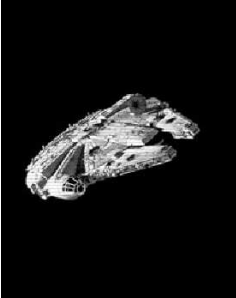
(b) $(f \cdot g)'(1)$

(c) $(f/g)'(1)$

39. Prove the Difference Rule.

40. Find all of the points on the graph $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1$ where the tangent line is horizontal.

41. Chewbacca is flying the Millenium Falcon along the curve $y = x^2$. If the Wookiee turns off his ion engines, the Falcon will fly off along a tangent line to this curve. He is almost out of power when he notices that a station on Yavin V is open at the point with coordinates $(5, 17)$. Making quick calculations, Chewie is able the determine the point (x, y) at which he should cut his engines (all Wookiees take calculus). What is the point he determined?



42. Find $x > 0$ so that the tangent lines to the graphs $y = x^2$ and $y = -x^2 + \frac{3}{2}x + 1$ at x are perpendicular.

Derivatives for Trigonometric Functions

43. Find $D_x y$.

(a) $y = 3 \sin x + 2 \cos x$

(b) $y = \sin^3 x$

(c) $y = \tan x + \sin x$

(d) $y = (\sin x)(\cos x)$

(e) $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

(f) $y = x \sin x$

(g) $y = x^3 \cos x$

(h) $y = \frac{x \sin x + \cos x}{x^2 + 1}$

(i) $y = \sin^2 x + \cos^2 x$

(j) $y = \cos x - x$

44. Use the definition of derivative to find $D_x(\cos 3x)$.

45. Find the equation of the tangent line to $y = \cos x$ at $x = \frac{\pi}{2}$.

46. A Ferris wheel of radius 10 meters is rotating counterclockwise at an angular velocity of 1 radian per second. At time $t = 0$, one seat on the wheel has coordinates $(10, 0)$.

(a) What are the coordinates of this seat at time t ?

(b) How fast is it rising (vertically) at time t ?

(c) When is it rising most quickly? What is its vertical speed at this time?

The Chain Rule

47. Find the derivative of each of the following functions:

(a) $f(x) = (x + 1)^{99}$

(b) $f(x) = \sin(x^2)$

(c) $w = (t^2 + 1)^{100}$

(d) $w = (t^3 + 1)^{100}$

(e) $f(x) = x \sin\left(\frac{1}{x}\right)$

(f) $g(t) = \tan(\cos t)$

(g) $f(x) = \sin^2 x$

(h) $h(t) = \frac{1}{\sin^2(t)}$

(i) $y = \sin^3(x^2 + x + 1)$

(j) $z = \cos^n x$, where n is a positive integer.

(k) $w = x^2 \sin(x^2)$

48. Given the following information:

$$f(2) = 1 \quad g(4) = 2$$

$$f(4) = 3 \quad g(3) = 4$$

$$f'(2) = 5 \quad g'(4) = 6$$

$$f'(4) = 7 \quad g'(3) = 8.$$

Find:

(a) $h(4)$ if $h(x) = f(g(x))$

(b) $h'(4)$ if $h(x) = f(g(x))$

(c) $h(4)$ if $h(x) = g(f(x))$

(d) $h'(4)$ if $h(x) = g(f(x))$

(e) $h'(4)$ if $h(x) = \frac{f(x)}{g(x)}$.

49. A pebble is dropped into still water, forming a circular ripple whose radius is expanding at a constant rate of 10 centimeters per second. Find a formula for the area enclosed by the ripple as a function of time. When the radius is 20 cm, how fast is the area enclosed by the ripple increasing?

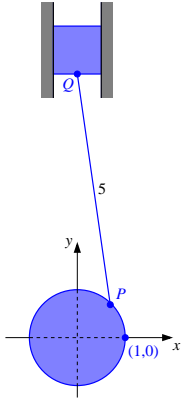
50. Use the Chain Rule to show that the derivative of an odd function is even and that the derivative of an even function is odd.

51. Consider the wheel-piston device in the figure below. The wheel has a radius of 1 foot and rotates counterclockwise at 2 radians per second. A point P is at $(1, 0)$ at time $t = 0$.

(a) Find the coordinates of P at time t .

(b) Find the y -coordinate of Q at time t (the x -coordinate is always zero).

(c) Find the velocity of Q at time t . You will need the fact that $D_u(\sqrt{u}) = \frac{1}{2\sqrt{u}}$.



Leibniz Notation

52. Use the Chain Rule to find $\frac{dy}{dx}$.

(a) $y = u^3$ and $u = x^2 + x + 1$

(b) $y = u^2$ and $u = \cos x$

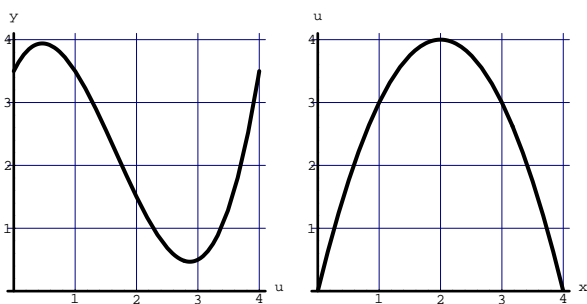
(c) $y = \sin\left(\frac{1}{x+1}\right)$

(d) $y = \sin^{57}(x^3 + 2x^2 + 1)$

(e) $y = \tan^2(\cos x)$

(f) $y = \cos[(x^2 + 1)^4]$

53. Consider the two graphs below.



Estimate $\frac{dy}{dx}$ for $x = 0, 1, 2, 3, 4$

54. A particular star about the size of our own sun explodes, forming a supernova. The shockwave from the event is a sphere of initial radius 10 kilometers. The radius is increasing at a rate of 40,000 kilometers per hour. How quickly is the volume enclosed by the shockwave increasing when the radius is 40,000 kilometers? 1,000,000 kilometers? (Your answer should be in cubic kilometers per hour).

55. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$ using the Product Rule and Chain Rule.

(b) Find $f'(x)$ for $x = 0$ using the definition of derivative.

(c) Show that $f'(x)$ is discontinuous at $x = 0$

56. Prove the Quotient Rule, using the Product Rule and the Chain Rule (this is, in fact, the way that I remember the Quotient Rule).

Higher-Order Derivatives

57. Find $\frac{d^3y}{dx^3}$

(a) $y = x^3 + 4x^4 + 3x + 5$

(b) $y = x^2 + 1$

(c) $y = \cos(3x)$

(d) $y = \sin(x^2)$

(e) $y = \frac{1}{x^2 + 1}$

58. Suppose a particle is moving horizontally along a line with position from the origin given by $s = 3t^3 - 5t^2 + t + 1$ where s is the directed distance in meters and t is the time in seconds.

(a) What are $v(t)$ and $a(t)$ the speed and acceleration of the particle at time t ?

(b) When is the particle moving toward the right (increasing s)?

(c) When is it moving toward the left (decreasing s)?

(d) When is the particle slowing down?

(e) Draw a picture to illustrate the motion of this particle. Hint: Use two dimensions; one for s and one for t . label interesting parts of your illustration.

59. A ball is thrown directly upwards. Its height in meters s above the ground t seconds later is given by $s = 20t - 5t^2$.

(a) What is the initial speed of the ball?

(b) When does it reach its maximum height?

(c) What is the maximum height attained?

(d) What is the speed of the ball and in which direction is it travelling at $t = 1$ second?

(e) When does the ball hit the ground?

(f) What is the speed of the ball immediately before it hits the ground?

Implicit Differentiation

60. Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $x^2 + y^2 = 1$

(b) $xy = 1$

(c) $x^2y^2 = x + 1$

(d) $\cos(xy) + \sin(xy) = \sqrt{2}$

(e) $x^2y + y^2x = 0$

61. For each of the equations in Problem 60, above, we give a point (x, y) which satisfies the equation. Find the equation of the tangent line to the corresponding curve at that point.

(a) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(b) $\left(\frac{1}{4}, 4\right)$

(c) $(1, \sqrt{2})$

(d) $\left(\frac{\pi}{4}, 1\right)$

(e) $(-1, 1)$

62. Find $\frac{dy}{dx}$.

(a) $y = \sqrt{x} - x^{9/5}$

(b) $y = x^{-2/3}$

(c) $y = \frac{1}{\sqrt{x^2 + 1}}$

63. The curve given by $x^2 + y^2 = 25$ is a circle of radius 5. Find y'' at $(4, 3)$ using implicit differentiation twice.

64. Suppose $s^3t + s^2t^2 = 1$. Find $\frac{ds}{dt}$ and $\frac{dt}{ds}$.

Related Rates

65. Each edge of a growing cube is increasing at a rate of 5 centimeters per second. How fast is the volume of the cube increasing when an edge is 20 cm long? How fast is the surface area increasing when an edge is 20 cm long?
66. A circular metal disk is heated and expands. Suppose the radius increases at a rate of 1 millimeter per second. How fast is the area of the disk expanding when its radius is 10 cm?
67. Sand is pouring from a pipe at the rate of 12 cubic feet per second. If the falling sand forms a conical pile on the ground whose height is always $\frac{1}{4}$ the diameter of the base, how fast is the height increasing when the pile is 4 feet high? *Hint:* Recall that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the height.
68. The hands on a clock are of length 5 inches (minute hand) and 4 inches (hour hand). How fast is the distance between the tips of the hands changing at 3:00?
69. Refer to Problem 54. Suppose that the radius R in kilometers of the shockwave t hours after the event is given by $R(t) = 10 + 40000t - 10t^2$.
 - (a) What is the initial radius of the shockwave?
 - (b) What is the initial speed of the shockwave?
 - (c) What is the initial volume enclosed by the shockwave?
 - (d) How fast is the volume enclosed by the shockwave growing initially?
 - (e) What is the speed of the shockwave at time $t = 1$ hour?
 - (f) How fast is the volume enclosed by the shockwave growing at time $t = 1$ hour?
 - (g) What is the speed of the shockwave at time $t = 24$ hours (1 day)?
 - (h) How fast is the volume enclosed by the shockwave growing at $t = 24$ hours?
 - (i) What is the volume of the shockwave at $t = 24$ hours?

Maxima and Minima

70. Identify the critical points and find the maximum value and the minimum value:

(a) $f(x) = x^2 + 2x + 2, -2 \leq x \leq 0.$

(b) $g(x) = x^2 - 3x + 2, -3 \leq x \leq 3.$

(c) $h(x) = \frac{1}{1+x}, 0 \leq x \leq 4.$

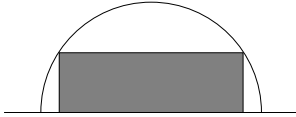
(d) $s(t) = t^{2/3}, -4 \leq t \leq 4.$

71. Find two positive numbers whose sum is 23 and whose product is a maximum.

72. Find two positive numbers a and b so that $a + b^2 = 136$ and so that $a^2 + b^2$ is a maximum.

73. Find the volume of the largest open box that can be made from a piece of cardboard with width 10 and length 12 by cutting squares from the corners and turning up the sides.

74. A rectangle is to be inscribed in a semicircle of radius r , as shown below. What are the dimensions of the rectangle if its area is to be maximized?



Monotonicity and Concavity

75. Determine where the graph of the given function is increasing, decreasing, concave up, and concave down. Then sketch the graph of the function.

(a) $f(x) = x^3 - x^2 - x + 1$.

(b) $g(x) = x^4 + x^2 + 1$.

(c) $h(x) = \frac{1}{x^2 + 1}$.

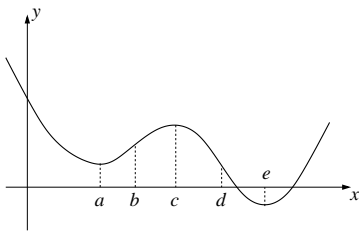
(d) $k(x) = \sin^2 x$ on $[0, 2\pi]$.

76. For the function f whose graph $y = f(x)$ is given below,

(a) Sketch $f'(x)$

(b) Where does $f'(x)$ change its sign?

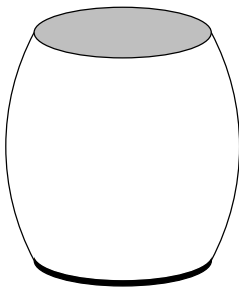
(c) Where does $f'(x)$ have a local extremum (i.e., maximum or minimum)?



77. Using Problem 76 as a guide, write a short paragraph (using complete sentences) which describes the relationship between the following features of a function f (an arbitrary function):

- The local extrema of f ,
- The points at which f changes concavity,
- The sign changes of f' ,
- The local extrema of f' ,

78. Water is flowing at a constant rate (i.e., constant volume flow per unit time) into the barrel shown below. Sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the widest point of the barrel.



Local Maxima and Minima

79. Find all critical points for the following functions, and determine the local extremal values (local maxima and local minima).

(a) $f(x) = x^3 - 4x$

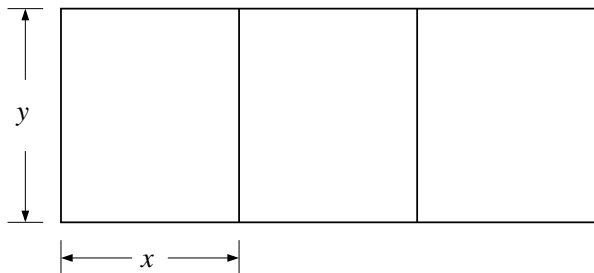
(b) $g(x) = x^4 + x^2 + 1$

(c) $s(t) = t + \frac{1}{t}, \quad t \neq 0$

80. Suppose f is a function with $f(c) = f'(c) = f''(c) = 0$ and $f'''(c) > 0$. What can you conclude about f ? Sketch a possible graph for f near c .

More Min-Max Problems

81. A farmer wishes to fence off three identical adjoining rectangular pens each with 16 square meters of area. What should the width, x , and length, y , of each pen be in order to minimize the amount of fence required?



82. Suppose that the outer boundary of the pens in Problem 81 requires heavy fence that costs \$10 per meter, but that the two internal partitions require fence which costs only \$5 per meter. What dimensions x and y yield the least expensive pens?
83. When you cough, your windpipe contracts. The velocity, v , with which air comes out depends on the radius, r , of your windpipe. If R is the normal (rest) radius of your windpipe, then for $r \leq R$, the velocity is given by

$$v = a(R - r)r^2,$$

where a is a constant. What value of r maximizes the velocity?

84. The temperature change, T , in a patient generated by a dose, D of a particular drug is given by

$$T = \left(\frac{C}{2} - \frac{D}{3} \right) D^2,$$

where C is a constant.

- (a) What dosage maximizes the temperature change?
- (b) The sensitivity of the body, at dosage D , to the drug is defined to be $\frac{dT}{dD}$. What dosage maximizes sensitivity?
85. Find the points on the ellipse

$$x^2 + 9y^2 = 9$$

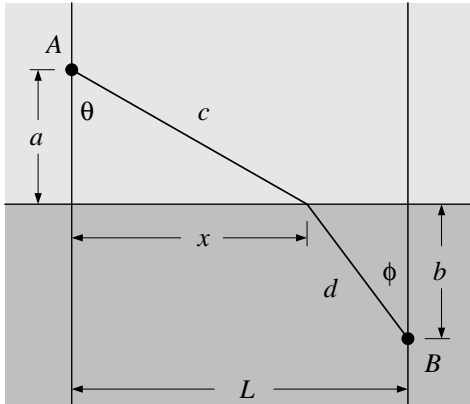
- (a) Closest to the point $(2, 0)$.
- (b) Closest to the focus $(\sqrt{8}, 0)$.

(Hint: Minimize the *square* of the distance. This avoids square roots.)

86. The illumination at a point is inversely proportional to the square of the distance of the point from the light source and directly proportional to the intensity of the light source. If two light sources are s feet apart and their intensities are I_1 and I_2 , respectively, at what point between them will the sum of their illuminations be a minimum?
87. A wire of length 20 cm is cut into two pieces; one is bent to form a square, and the other is bent to form an equilateral triangle. Where should the cut be made if
- (a) the sum of the two areas is to be a minimum?
- (b) a maximum?
- (Allow the possibility of not cutting the wire at all.)

88. An observatory is to be in the form of a right circular cylinder surmounted by a hemispherical dome. If the hemispherical dome costs twice as much per square foot as the cylindrical wall, what are the most economical proportions for a given volume?
89. Fermat's Principle in optics says that light travels from point A to point B along the path that requires least time. Suppose that light travels in the top medium at velocity c and in the bottom medium at velocity d . If θ and ϕ are the angles the of light make with the vertical, show that

$$\frac{\sin \theta}{c} = \frac{\sin \phi}{d}.$$



Sophisticated Graphing

90. Analyze the following functions and sketch their graphs:

(a) $f(x) = x^3 - 2x^2 - 3x + 1$

(b) $g(x) = (x + 1)^4$

(c) $h(x) = \frac{x^2 - x + 3}{x - 1}$

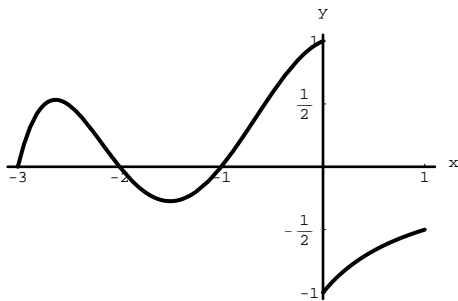
(d) $s(t) = \frac{t^2}{t^2 + 4}$

91. Sketch the graph of the function f that has the following properties:

- f is everywhere continuous,
- $f(-2) = 3, f(0) = 0, f(3) = 2,$
- $f'(-2) = 0, f'(3) = 0, f'(x) < 0$ for $x < -2, f'(x) < 0$ for $-2 < x < 0, f'(x) > 0$ for $0 < x < 3,$
 $f'(x) < 0$ for $x > 3,$
- $f''(-2) = f''(-1) = f''(2) = 0,$
- $f''(x) > 0$ for $x < -2$ and for $-1 < x < 2,$
- $f''(x) < 0$ for $-2 < x < -1$ and for $2 < x.$

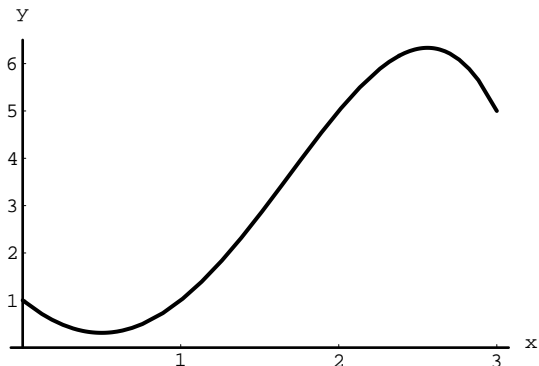
92. Suppose $f'(x) = (x - 2)(x - 1)^2(x + 1)$ and $f(1) = 0$. Sketch a possible graph for f .

93. Let f be a continuous function with $f(0) = f(1) = 0$. If the graph of $y = f'(x)$ is as shown below, sketch a possible graph for $y = f(x)$



The Mean Value Theorem

94. Prove **Rolle's Theorem**: If f is continuous on $[a, b]$ and differentiable on (a, b) , and if $f(a) = f(b)$, then there is at least one number c in (a, b) so that $f'(c) = 0$.
95. For the function graphed below, find (approximately) all points c that satisfy the conclusion to the Mean Value Theorem for the interval $[0, 3]$.



96. Prove that if $F'(x) = 0$ for all x in (a, b) then there is a constant C such that $F(x) = C$ for all x in (a, b) . *Hint*: Let $G(x) = 0$ and apply the Antiderivative Theorem.
97. Suppose we have two functions $f(x)$ and $g(x)$ such that $f(0) = k$ and $g(0) = 0$, $f'(x) = -g(x)$ and $g'(x) = f(x)$. Show that $[f(x)]^2 + [g(x)]^2 = k^2$. *Hint*: Let $F(x) = [f(x)]^2 + [g(x)]^2$ and use Problem 96. Use this result to prove that $\cos^2 x + \sin^2 x = 1$.
98. Use the Mean Value Theorem to show that
- $$|\sin x - \sin y| \leq |x - y|.$$
99. Suppose we are watching a horse race between Abel's Fortune, and Buttercup. If the two horses start at the same point and finish in a dead heat, prove that their speeds were identical at some instant of the race. (*Hint*: Let $s(t)$ be the distance between the two horses at time t and use Rolle's Theorem).
100. Renzo is driving his car. He stops at a toll booth. Twenty minutes later at a point 25 miles down the road, Renzo is clocked by a police officer at 60 miles per hour. If the speed limit is 60 miles per hour, prove to the officer that Renzo was speeding somewhere between the toll booth and the officer.
101. Is it possible for a function f to have all of the following properties?
- f is continuous on $[0, 1]$,
 - f is differentiable on $(0, 1)$,
 - f has a tangent line at every point in $[0, 1]$, and
 - f is not differentiable on $[0, 1]$.

Either give an example of such a function, or explain why this is impossible.

Antiderivatives

102. Find the indicated antiderivatives.

(a) $\int 2 \, dx.$

(b) $\int x + \pi \, dx.$

(c) $\int x^{1000} + x^2 \, dx.$

(d) $\int \frac{x^5 - 2x}{x^3} \, dx.$

(e) $\int \theta + \cos \theta \, d\theta.$

(f) $\int 4x^{5/2} \, dx.$

(g) $\int (\sqrt{3}x^2 + 1)^{11} 2\sqrt{3}x \, dx.$

(h) $\int (9x^2 - 2x)(6x^3 - 2x^2 + 1)^3 \, dx.$

(i) $\int \cos^3 \left(2\pi x + \frac{\pi}{6} \right) \sin \left(2\pi x + \frac{\pi}{6} \right) \, dx.$

(j) $\int 3x \sin(x^2) \, dx.$

103. In each of the following problems $f''(x)$ is given. Find $f(x)$ by antidifferentiating twice. Note that $f(x)$ should be a family of functions given by two parameters C_1 and C_2 , one from each antiderivative.

(a) $f''(x) = x$

(b) $f''(x) = -9.81$

(c) $f''(x) = \frac{x^5 + x}{x^3}$

(d) $f''(x) = \sin x$

104. Prove the formula

$$\int [f(x)g'(x) + f'(x)g(x)] \, dx = f(x)g(x) + C.$$

(Hint: How did we prove all of the formulas in class?)

105. Referring to Problem 104, What other interesting formulas for antiderivatives can you come up with and prove? (Hint: Think about what rules we have for derivatives).

106. Find $\int |x| \, dx$. (Hint: Write down the definition for $|x|$, then recall what $\int f(x) \, dx$ actually means.)

Introduction to Differential Equations

107. For each of the following differential equations, find the general solution (the family of solutions parameterized by the constant C). Then find a particular solution which also satisfies the given condition.

(a) $\frac{dy}{dx} = x^2 + x$, $y = 1$ at $x = 1$

(b) $\frac{dy}{dx} = \frac{x}{y}$, $y = 2$ at $x = -2$

(c) $\frac{dy}{dx} = -\frac{x}{y}$, $y = 3$ at $x = 4$

(d) $\frac{dy}{dx} = \frac{\sin x}{\cos y}$, $y = \frac{\pi}{2}$ at $x = 0$

(e) $\frac{dy}{dt} = y^2(t^2 - 1)$, $y = 2$ at $t = 0$

(f) $\frac{d^2s}{dt^2} = s^3$, $s = 2$ at $t = 1$, $\frac{ds}{dt} = 1$ at $t = 1$

108. Find the xy -equation for the curve through $(0, 1)$ whose slope at any point is twice the square of its y -coordinate.

109. A projectile is thrown upward from a height s_0 above the surface of the Earth with a speed of v_0 . Let $s(t)$ be its height above the surface of the Earth at time t . Let g be the acceleration due to gravity. Write down a differential equation relating the acceleration of the projectile to acceleration due to gravity. Solve this differential equation with the initial conditions $s(0) = s_0$, and $\frac{ds}{dt}(0) = v_0$ to get the familiar formula from physics

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$$

for the height s at time t .

110. Refer to Problem 109. A ball is thrown directly upwards from the ground with an initial speed of 20 meters per second. What is the maximum height the ball reaches? (Use the approximate value for the acceleration due to gravity $g = 9.81\text{m/sec}^2$).

111. A certain rocket, initially at rest, is shot straight up with an acceleration of $6t$ meters per second per second during the first 10 seconds after blast-off, after which the engine cuts out and the rocket is subject only to gravitation acceleration of -10 meters per second per second. How high will the rocket go? (You may assume uniform gravity.)

112. The wolf population P in a certain state has been growing at a rate proportional to the cube root of the population size. the population was estimated at 1,000 in 1990 and at 1,800 in 2000.

(a) Write the differential equation for P at time t with the two corresponding conditions (take the year 2000 to be time $t = 0$).

(b) Solve the differential equation.

(c) What do you expect the population to be in 2010?

(d) When will the population reach 4,000 animals?

113. The rate of change of volume V of a melting snowball is proportional to the surface area S . That is

$$\frac{dV}{dt} = -kS$$

where k is a positive constant. Rewrite this differential equation as a differential equation in r , the radius, and t . Solve this new equation (your solution should have two unknown constants, k and the constant of integration C).

Sums and Sigma Notation

114. Find the value of each of the following sums.

(a) $\sum_{n=1}^5 (n+1)$

(b) $\sum_{i=1}^7 i^2$

(c) $\sum_{k=0}^5 \frac{(-1)^k 2^k}{k+1}$

(d) $\sum_{n=0}^4 \frac{3^n}{n!}$, where $0! = 1$ and $n! = n[(n-1)!]$.

(e) $\sum_{n=0}^5 n \cos(n\pi)$

115. Write each of the following summations in sigma notation.

(a) $1 + 2 + 3 + \cdots + 27$

(b) $1 + 4 + 9 + 16 + \cdots + 625$

(c) $a_1 + \cdots + a_N$

(d) $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

(e) $b_1 + b_3 + b_5 + \cdots + b_{101}$

(f) $f(c_1) + f(c_2) + f(c_3) + \cdots + f(c_N)$

(g) $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_N)\Delta x$

116. Suppose that $\sum_{i=1}^N a_i = A$ and $\sum_{i=1}^N b_i = B$. Find

(a) $\sum_{i=1}^N (a_i - b_i)$

(b) $\sum_{i=1}^N (2a_i + 3b_i)$

(c) $\sum_{i=1}^N (a_i + b_i + 1)$

117. Find the value of each of the following sums. (Hint: They collapse!)

(a) $\sum_{k=1}^{23} \left(\frac{1}{k} - \frac{1}{k+1} \right)$

(b) $\sum_{k=1}^{10} (A^k - A^{k-1})$, where A is some constant.

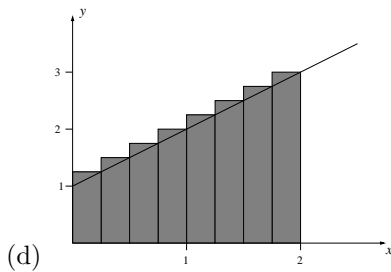
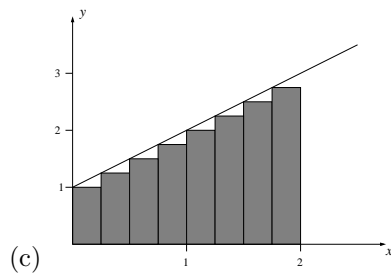
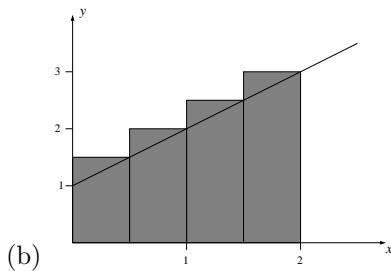
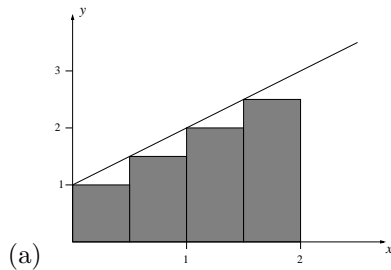
118. Prove the following formula for a **geometric sum**

$$\sum_{k=0}^N r^k = 1 + r + r^2 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r}$$

(Hint: Let $S = 1 + r + r^2 + \cdots + r^N$. Evaluate $S - rS$ and solve for S .)

Introduction to Area

119. Find the area of the indicated polygons on the graph $y = x + 1$ between $x = 0$ and $x = 2$.



120. In Problem 119, we had either four or eight rectangles either inscribed or circumscribed on the graph of $y = x + 1$ between $x = 0$ and $x = 2$. What would the area be if we had:

- 16 rectangles inscribed on the graph $y = x + 1$ between $x = 0$ and $x = 2$?
- 16 rectangles circumscribed on the graph $y = x + 1$ between $x = 0$ and $x = 2$?
- 2^n rectangles inscribed on the graph $y = x + 1$ between $x = 0$ and $x = 2$?
- 2^n rectangles circumscribed on the graph $y = x + 1$ between $x = 0$ and $x = 2$?
- What happens as $n \rightarrow \infty$?

121. Repeat Problem 120, but with the graph $y = x^2 + 1$ between $x = 0$ and $x = 2$.

122. Suppose that an object is traveling in a straight line, and its velocity at time t seconds is given by $v(t) = t + 1$. How far did the object travel between $t = 0$ and $t = 2$?

The Definite Integral

For this section, you may use the Fundamental Theorems of Calculus to **check** your answers, but you should use only the definition of Definite Integrals to calculate.

123. For each function f , interval I , and partition φ with sample points, calculate the Riemann sum

$$\sum_{i=1}^n f(\bar{x}_i) \Delta x_i.$$

- (a) $f(x) = x^2 + 1$, $I = [0, 2]$, φ : divide I into 10 subintervals, evaluate at the midpoint of each subinterval.
- (b) $f(x) = x^3$, $I = [0, 1]$, φ : divide I into n subintervals, evaluate at the left-hand point of each subinterval. Is this a lower bound or an upper bound on the integral?
- (c) $f(x) = x^3$, $I = [0, 1]$, φ : divide I into n subintervals, evaluate at the right-hand point of each subinterval. Is this a lower bound or an upper bound on the integral?

124. Use the formula

$$\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

and Problem 123 to calculate $\int_0^1 x^3 dx$.

125. Find $\int_0^4 x + 2 dx$ using the definition.

126. Recall that $[[x]]$ denotes the greatest integer less than or equal to x . Find

- (a) $\int_{-1}^3 [[x]] dx$
- (b) $\int_{-1}^3 [[x]]^2 dx$
- (c) $\int_{-1}^3 [[x^2]] dx$

127. Let f be an odd function and g be an even function. Furthermore, suppose

$$\int_0^2 |f(x)| dx = \int_0^2 g(x) dx = 5.$$

Use (and state) geometric reasoning to calculate each of the following:

- (a) $\int_{-2}^2 f(x) dx$
- (b) $\int_{-2}^2 g(x) dx$
- (c) $\int_{-2}^2 |f(x)| dx$
- (d) $\int_{-2}^2 f(x)g(x) dx$

128. State whether or not you think the following functions are integrable on every interval $[a, b]$. Justify your claim.

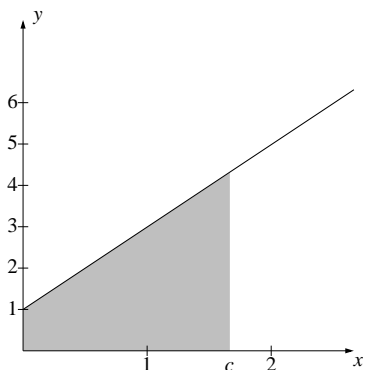
(a) $f(x) = x^{57} + x^{32} - 122372x + 1$

(b) $f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer,} \\ x, & \text{otherwise.} \end{cases}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{otherwise.} \end{cases}$

The First Fundamental Theorem of Calculus

129. State the First Fundamental Theorem of Calculus. Explain, in ordinary English, what it means using speed and position of a car as analogy.
130. We will look at a special kind of function, called an **accumulation function**. You probably saw such functions in Section 0.5 while reviewing at the beginning of the semester. Let $A(c)$ denote the area of the region bounded from above by the line $y = 2x + 1$, from the left by the y -axis, from below by the x -axis, and from the right by the line $x = c$.



- (a) Write $A(x)$ as an integral formula.
- (b) Graph $A(x)$.
- (c) Graph $A'(x)$.
131. Suppose that $\int_0^1 f(x) dx = 3$, $\int_1^3 f(x) dx = 5$, $\int_0^1 g(x) dx = 7$, $\int_1^3 g(x) dx = -2$. Use the properties of definite integrals to find each of the following. State which properties you are using.
- (a) $\int_1^3 3f(x) dx$
- (b) $\int_0^3 3f(x) dx$
- (c) $\int_0^3 [3f(x) + 2g(x)] dx$
- (d) $\int_1^0 f(x) dx$
- (e) $\int_2^2 g(x) dx$
- (f) $\int_0^3 [3f(x) + 2g(x) + 2\pi] dx$

132. Find $G'(x)$

(a) $G(x) = \int_0^x 3t \, dt$

(b) $G(x) = \int_1^x 3t \, dt$

(c) $G(x) = \int_x^2 t^2 + 2 \, dt$

(d) $G(x) = \int_1^x \cos^{57}(t) + t \, dt$

(e) $G(x) = \int_{x^3}^1 t \, dt$

(f) $G(x) = \int_1^x xt \, dt$ (Hint: This one is not as easy as it seems).

(g) $G(x) = \int_x^{x^2} \sin t \, dt$

(h) $G(x) = \int_{x^2}^{\cos x} t^3 \, dt$

133. Find the interval on which the graph of $y = f(x)$ is concave up if

$$f(x) = \int_1^x \frac{t}{t^2 + 1} \, dt.$$

134. Let $G(x) = \int_0^x \sin t \, dt$.

(a) Find $G(0)$ and $G(2\pi)$. (Hint: this should be easy if you think about it)

(b) Let $y = G(x)$. Then the First Fundamental Theorem of Calculus says that $\frac{dy}{dx} = G'(x) = \sin x$.

Solve the differential equation $\frac{dy}{dx} = \sin x$.

(c) Find the particular solution to this differential equation that satisfies $y = G(0)$ when $x = 0$. Why do we want this particular solution?

(d) Show that $\int_0^\pi \sin x \, dx = 2$.

135. Does there exist a function f such that $\int_0^x f(t) \, dt = x^2 + 1$? Explain.

The Second Fundamental Theorem of Calculus

136. Use the Second Fundamental Theorem of Calculus to evaluate each of the following integrals.

(a) $\int_1^2 x \, dx$

(b) $\int_2^3 x^2 \, dx$

(c) $\int_0^{2\pi} \theta + \cos \theta \, d\theta$

(d) $\int_0^1 x^{1000} + x^2 \, dx$

(e) $\int_2^3 3x^{5/2} \, dx$

(f) $\int_4^5 (\sqrt{3}x^2 + 1)^{11} 2\sqrt{3}x \, dx$

(g) $\int_0^1 (9x^2 - 2x)(6x^3 - 2x^2 + 1)^3 \, dx$

(h) $\int_{\pi}^{2\pi} 4x \sin(x^2) \, dx$

(i) $\int_0^{\pi} \cos^2 x \sin x \, dx$

(j) $\int_0^{x^2} 3t^3 \, dt.$

137. Find the average value of each of the following functions on the given interval.

(a) $f(x) = 3x^2; \quad [1, 3]$

(b) $f(x) = 2 \cos x + 1; \quad [0, \pi]$

138. Show that $F(x) = x|x|$ is an antiderivative for $f(x) = 2|x|$. Use this to find a simple formula for

$$\int_a^b 2|x| \, dx.$$

139. Give an example to show that the accumulations function $G(x) = \int_a^x f(t) \, dt$ can be continuous even if f is not continuous.

Evaluating Definite Integrals

140. Evaluate.

(a) $\int_0^2 \sqrt{3t-2} dt$

(b) $\int_1^4 x\sqrt{2x^2+1} dx$

(c) $\int_0^1 \frac{x^3}{(x^4-2)^{3/2}} dx$

(d) $\int_{\pi/4}^{\pi/3} \frac{\sin \theta}{\cos^5 \theta} d\theta$

(e) $\int_{-1}^1 3x^2 \sin^3(x^3) \cos(x^3) dx$

(f) $\int_1^2 \left(1 - \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt$

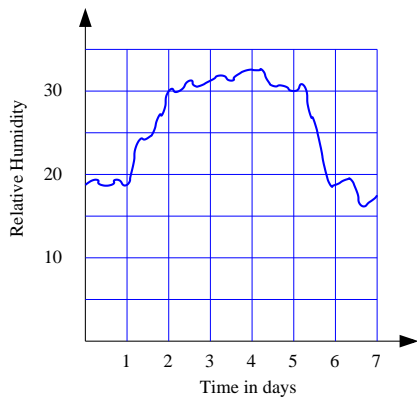
(g) $\int_{-\pi}^{\pi} \theta^3 \cos \theta d\theta$

(h) $\int_{-1}^1 (|x^5| + x^5) dx$

141. Let f be a continuous function. Prove that $\int_a^b f(-x) dx = -\int_{-a}^{-b} f(x) dx$. (Hint: substitution).

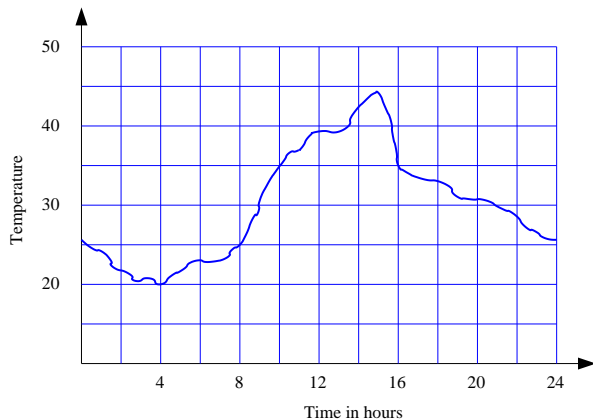
Mean Value Theorem for Integrals and the Use of Symmetry

142. The graph below shows the relative humidity H as a function of time t (measured in days since Sunday) for an office building. Approximate the average relative humidity for the week.



143. The graph below shows the temperature T as a function of time t (measured in hours past midnight) for one day in Salt Lake City.

- (a) Approximate the average temperature for the day.
 (b) Must there be a time when the temperature is equal to the average temperature for the day? Explain.



144. Use symmetry to evaluate the following integrals:

- (a) $\int_{-\pi}^{\pi} (\sin x + \cos x) dx$
 (b) $\int_{-1}^1 \frac{x^3}{(1+x^8)} dx$
 (c) $\int_{-\pi/3}^{\pi/3} \frac{\sin^3 x}{1+\cos x} dx$
 (d) $\int_{-\pi/2}^{\pi/2} (\theta + \sin \theta + \theta \cos \theta + \sin^3 \theta) d\theta$

145. Calculate $\int_0^{4\pi} |\sin x| dx$.

The Area of a Plane Region

146. For each of the following problems, sketch the region R bounded by the graphs of the given equations, and set up the integral which gives the area of R . Make a reasonable guess as to the area of R (approximate with other shapes).
- (a) $y = x^2 + 1$, and $y = 0$, between $x = 0$ and $x = 2$
 - (b) $y = (x - 2)(x - 3)$, and $y = 0$ between $x = 2$ and $x = 3$.
 - (c) $y = 4x^2$ and $y = 8 - 4x^2$.
 - (d) $3x = y^3$ and $y = 3x$
147. Evaluate the integrals you obtained in Problem 146
148. Consider the curve $y = \frac{1}{x^3}$ on the interval $[1, 2]$
- (a) Find the area under this curve on this interval.
 - (b) Determine the number c so that the line $x = c$ bisects the area of part (a).
 - (c) Determine the number d so that at the point $x = d$, the curve obtains the average height on the interval $[1, 2]$ (Hint: see the Mean Value Theorem for Definite Integrals).
149. Bonaventura Cavalieri (1598-1647) developed the famous Cavalieri Principle in 1635: Let R and S be two regions in the plane situated above the interval $[a, b]$. Suppose that at each point x in the interval $[a, b]$ the heights of the two regions are the same. Then R and S have the same area.
- (a) Draw a picture which illustrates the principle.
 - (b) State in one or two sentences why this should be true.
 - (c) Prove that it is true.

Volumes of Solids

150. For each of the problems below, a planar region R is given. Find the volume of the solid W_x obtained by revolving R about the x -axis. Then find the volume of the solid W_y obtained by revolving R about the y -axis.

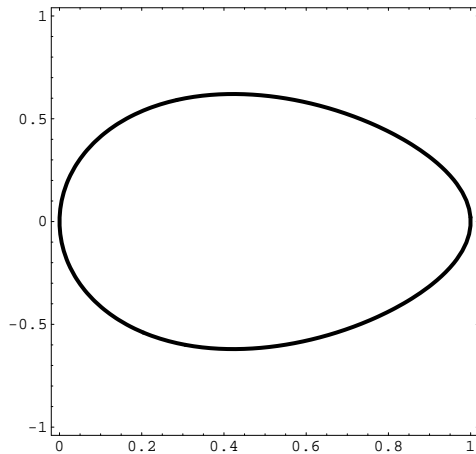
- (a) R is the region bounded by $y = x^2 + 1$, the line $x = 1$ and the coordinate axes.
- (b) R is the region bounded by $y = x - x^2$ and the x -axis.
- (c) R is the region bounded by $y = 1 - x^2$ and the coordinate axes.
- (d) R is the triangle with vertices at the origin, at $(0, 2)$, and at $(3, 0)$.
- (e) R is the region bounded by $y = \frac{1}{x}$, the lines $x = 1$ and $x = 2$, and the x -axis.
- (f) R is the region bounded by $x = y^2$, and the horizontal lines $y = 1$ and $y = 2$, and the y -axis.
- (g) R is the region bounded by the line $y = 2x$ and the parabola $y = 2x^2$.

151. Find the volume of the *ellipsoid* formed by revolving the upper half of the ellipse

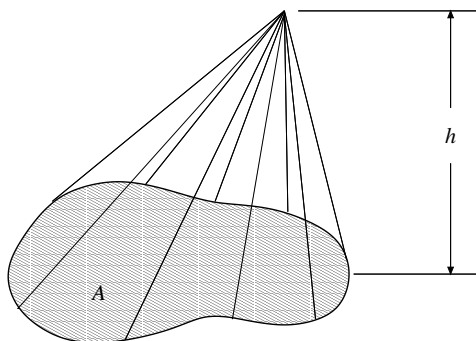
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x -axis. Assume that $a > b > 0$.

152. Below is a portion of the graph of $y^2 = x(x-1)(x-2)$. Let R be the upper half of this region bounded by this curve, and let W be the egg-shaped solid formed by revolving R about the x -axis. Find the volume of W .



153. Show that the volume of a general cone (shown below) is $\frac{1}{3}Ah$ where A is the area of the base and h is the height.



154. State the version of Cavalieri's Principle for volume (see Problem 149).

Length of a Plane Curve; Surface Area

155. Sketch each of the following curves and find their lengths.

(a) $y = 9x^{3/2}$ between $x = 1$ and $x = 4$

(b) $y = \frac{x^4 + 1}{2x}$ between $x = 1$ and $x = 2$

(c) $2xy^3 - y^4 = 1$ between $y = 1$ and $y = 2$

(d) $x = t^2 + 1, \quad y = t^3 + 2t, \quad 0 \leq t \leq 2$

(e) $x = 2 \sin t - 1, \quad y = 2 \cos t + 3, \quad 0 \leq t \leq \frac{\pi}{4}$.

156. The curve in Problem 155(e) is revolved around the x -axis. What is the area of the resulting surface?

157. A point P on the rim of a wheel of radius a is initially at the origin. As the wheel rolls to the right along the x -axis, P traces out a curve called a **cycloid** (I encourage you to set up a simple experiment at home to see what this curve looks like). The cycloid is given by the parametric equations $x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta)$ where the parameter θ is the angle through which the wheel has rolled.

Find the length of one arch of the cycloid, by first showing that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4a^2 \sin^2\left(\frac{\theta}{2}\right).$$

158. Suppose that the wheel of Problem 157 turns at a constant rate $\omega = \frac{d\theta}{dt}$, where t is time. Then $\theta = \omega t$.

(a) Show that the speed $\frac{ds}{dt}$ of P along the cycloid is

$$\frac{ds}{dt} = 2a\omega \left| \sin \frac{\omega t}{2} \right|.$$

(b) When is the speed a maximum and when is it a minimum?

(c) Explain why a bug on a wheel of a car going 60 miles per hour is itself sometimes traveling at 120 miles per hour.

159. Find the area of the surface generated by revolving the given curve about the x -axis.

(a) $y = \sqrt{16 - x^2}, \quad -1 \leq x \leq 2$

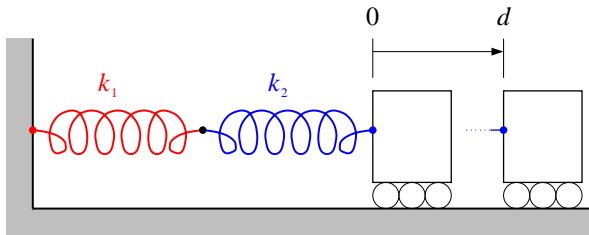
(b) $y = \frac{1}{3}x^3, \quad 1 \leq x \leq 2$

(c) $x = 2t, y = t^3, \quad 0 \leq t \leq 1$

(d) $x = 1 - t^2, y = t, \quad 0 \leq t \leq 1$

Work and Fluid Force

160. A force of 5 pounds is required to keep a spring stretched 2 feet beyond its normal length.
- Find the value for the spring constant.
 - Find the work done in stretching the spring 2 feet beyond its natural length.
 - Find the work done in compressing the spring $1/2$ foot.
161. For any spring obeying Hooke's Law, show that the work done in stretching it a distance d is given by $W = \frac{1}{2}kd^2$.
162. For a certain type of nonlinear spring, the force required to keep the spring stretched a distance s is given by the formula $F = ks^{4/3}$. If the force required to keep it stretched 10 cm is 0.25 N, how much work is done in stretching this spring 30 cm?
163. Two springs, each obeying Hooke's Law, are attached in series as shown in the figure below. The left-hand spring has spring constant k_1 and the right-hand spring has spring constant k_2 . The sliding block is displaced a distance of d to the right, stretching both springs.



- How much force is required to displace the sliding block?
 - How far is each spring stretched?
 - How much work was required to displace the block?
164. One cubic foot of gas under a pressure of 80 pounds per square inch expands adiabatically to 4 cubic feet according to the law $pv^{1.4} = k$, where k is a constant. Find the work done by the gas.
165. According to Coulomb's Law, two like electrical charges repel each other with a force that is inversely proportional to the square of the distance between them. If the force of repulsion is 10 dynes (1 dyne = 10^{-5} Newtons) when they are 2 centimeters apart, find the work done in bringing the charges from 5 centimeters apart to 1 centimeter apart.

Moments and Center of Mass

166. John and Mary, weighing 180 and 110 pounds, respectively, sit at opposite ends of a 12-foot see-saw with the fulcrum in the middle. Where should their 80-pound son Tom sit in order for the board to balance?
167. A straight wire 7 units long has density $\delta(x) = 1 + \sqrt{x}$ at a point x units from one end. Find the distance from this end to the center of mass.
168. Find the centroid of the region bounded by the given curves. Make a sketch and use symmetry where possible.
- (a) $y = 2 - x$, $y = 0$, $x = 0$
 - (b) $y = \frac{1}{3}x^2$, $y = 0$, $x = 4$
 - (c) $y = x^2$, $y = x + 3$