# 1 Free groups

We start with a (usually finite) set $S$, called *alphabet*. For example, $S$ could be $\{a, b\}$. We need symbols for *inverses* of these letters, and we could use $a^{-1}$ and $b^{-1}$, but it's shorter to use capital letters $A$ and $B$.

A *word* in $S$ is any finite sequence in $S \cup S^{-1} = \{a, b, A, B\}$, e.g. $aba$ or $aBaABAa$. The empty word is also allowed, and we'll denote it by 1.

If you have two words, e.g. $v = aba$ and $w = aBa$, you can concatenate them and get a new word. This can be done either as $vw = abaaBa$ or as $wv = aBaaba$, and generally, $vw \neq wv$.

We'll also agree to *cancel* inverse symbols, i.e. remove subwords of the form $aA, Aa, bB, Bb$. For example, $aAb = b$ and $abBA = 1$. The latter example requires two cancelations.

When no cancelations are possible, we say the word is *reduced*. Every word can be transformed to a reduced word by a sequence of cancelations.

The set of all reduced words is denoted by $F$, or if one wants to be explicit about the starting alphabet, by $F(S)$. It is a group under the operation "concatenate and reduce". For example,

$$ab \cdot Ba = aa, bA \cdot aB = 1$$

It is somewhat nontrivial to check that this operation is associative. You can look this up in Massey or Hatcher, and we can discuss it in class. More generally, *no matter how you reduce a word, you always get the same reduced word.*

*Question* 1.1. Why is this more general than associativity?

*Exercise* 1.2. Check the other group axioms. The identity element is 1, and the inverse is obtained replacing each letter by its inverse and reversing the order of all letters. E.g. $(abAb)^{-1} = BaBA$.

*Exercise* 1.3. Draw an infinite 4-valent tree (well, a big part of it) with each edge either horizontal or vertical. Label a vertex 1 and all horizontal edges are labeled $a$ and oriented from left to right. All vertical edges are labeled $b$ and oriented from bottom to top. Now any vertex is connected by a path from 1 and it is assigned the reduced word obtained by reading off the labels on the edges. If an edge is traversed in the wrong direction, the associated label is reversed. Every reduced word is the label of exactly one vertex. Thus this tree (called the *Cayley tree*) is the "picture" of the free group. Can you prove associativity (or the more general unique reduced word claim) using the tree?

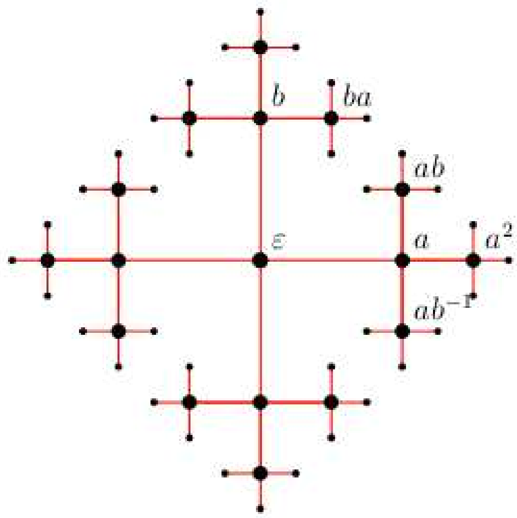


Figure 1: Cayley tree, from Löh: *Geometric Group Theory, an Introduction*

*Exercise* 1.4. Suppose the alphabet has at least two letters. Show that $F$ is not commutative. Also show that the center of the group

$$Z(F) = \{g \in F \mid gx = xg \text{ for all } x \in F\}$$

is the trivial subgroup. Which elements commute with a letter from $S$? Which elements commute with a word like $aabA$?

## 2 (Combinatorial) Fundamental Group

This is a group associated to any graph one of whose vertices is distinguished, and called the *basepoint*. When the graph is a *rose*, i.e. a collection of circles all meeting in one point, the associated group is just the free group we discussed above. In general, the group will also be isomorphic to a free group on a suitable alphabet.

So let $\Gamma$ be a graph. We will associate letters like $a, b, \cdots$ to the edges of $\Gamma$ and orient each edge. When we mean edge $a$ but with opposite orientation, we call it $a^{-1}$ or $A$.

An *edge path* is a sequence of oriented edges such as $abcA$ such that each begins where the previous ends. But $aB$ is *not* an edge path, see Figure below.

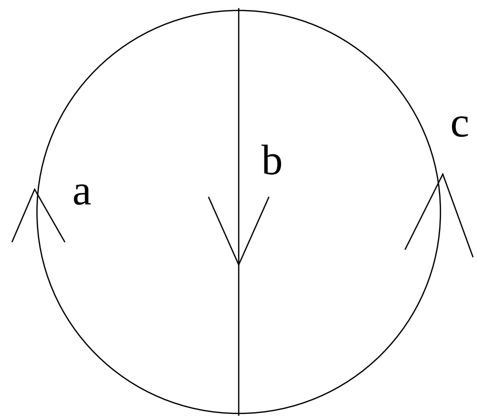


Figure 2: A graph with labeled edges

An edge path is allowed to backtrack, e.g. $aAc$, and as in the case of words, we are allowed to *cancel* or *reduce*, and get a *homotopic* edge path. E.g. $aAc \sim c$.

Edge paths can sometimes be concatenated, when the first ends where the second begins.

Now say our graph is equipped with a basepoint, for example the initial vertex of the edge $a$. We consider only edge paths that start and end at the basepoint. Such edge paths are sometimes called *closed* edge paths, or *circuits*. These paths can always be concatenated, and the set of reduced circuits forms a group under the "concatenate and reduce" operation. This group is called the *fundamental group* of the graph, with respect to the selected basepoint.

It turns out that if you choose a different basepoint, you still get a group isomorphic to the one you had before (as long as the graph is connected). It also turns out that the fundamental group of the graph is always isomorphic to a free group. We will discuss this in class, but you may want to construct an explicit isomorphism between the fundamental group in the above example and the free group $F(\{a, b\})$. Hint: Just drop $c$!