1 $SL_2(\mathbb{R})$ problems

1. Show that $SL_2(\mathbb{R})$ is homeomorphic to $S^1 \times \mathbb{R}^2$.

   Hint 1: $SL_2(\mathbb{R})$ acts on $\mathbb{R}^2 - 0$ transitively with stabilizer of $(1, 0)$ equal to the set of upper triangular matrices with 1's on the diagonal, which is homeomorphic to $\mathbb{R}$. This gives a "principal bundle", i.e. a map $SL_2(\mathbb{R}) \to \mathbb{R}^2 - 0$ whose point inverses are lines. Construct a section $\mathbb{R}^2 - 0 \to SL_2(\mathbb{R})$ and then a homeomorphism to the product.

   Hint 2: Same thing, but for the action of $SL_2(\mathbb{R})$ on the upper half plane, with point inverses circles. Better yet (but uses more sophisticated math): $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\pm I$ can be identified with the unit tangent bundle $T_1\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$, which in turn can be identified with $\mathbb{H}^2 \times S^1_\infty$, product with the circle at infinity. This shows that $PSL_2(\mathbb{R})$ is homeomorphic to $S^1 \times \mathbb{R}^2$, but $SL_2(\mathbb{R})$ is a double cover of $PSL_2(\mathbb{R})$.

2. We have seen that trace $Tr(A)$ determines the conjugacy class of $A \in SL_2(\mathbb{R})$ provided $|Tr(A)| > 2$. Show that this conjugacy class, as a subset of $SL_2(\mathbb{R})$, is a closed subset homeomorphic to $S^1 \times \mathbb{R}$.

3. The set of parabolics is not closed and consists of four conjugacy classes (two with trace 2 and two with trace -2), each homeomorphic to $S^1 \times \mathbb{R}$.

4. There are two conjugacy classes of elements of $SL_2(\mathbb{R})$ whose trace is a given number in $(-2, 2)$, each closed and homeomorphic to $\mathbb{R}^2$. If A is such a matrix, how can you tell (quickly) whether it is a clockwise or a counterclockwise rotation? Hint: For a counterclockwise rotation, the determinant of the matrix with columns v and Av is positive. Then try $v = e_1$ or $v = e_2$.

5. (Harder) Consider an Anosov homeomorphism $f : T^2 \to T^2$. Show that the set of periodic points is dense. Can you estimate the number of fixed points of f^k, for large k? We will discuss these questions later in the course.

2 Dynamics problems

6. Show that the definition of entropy does not depend on the choice of the metric.

7. More generally, show that if there is a semiconjugacy from $f : X \to X$ to $g : Y \to Y$ then $h(f) \geq h(g)$. This corresponds to the intuition that

entropy measures dynamical complexity, and collapsing leads to simplification. Hint for both 1 and 2: Uniform continuity.

8. Compute the entropy of the identity on X, of a rotation on S^1, more generally of an isometry. (Show it is 0. The intuition is that isometries are dynamically very simple.)

9. Compute the entropy of a subshift of finite type.

10. (Harder) For an Anosov homeomorphism f of the torus with dilatation λ prove that $h(f) \geq \log \lambda$.

    Hint: For a small ϵ consider a set of points in the torus that's arranged roughly as a square grid with sidelengths $\sim \epsilon/\lambda^k$. There are $\sim \lambda^{2k}/\epsilon^2$ points and any two are distinguishable by f^i for some $i = -k, \cdots, k$. The same proof works for pseudoAnosov homeomorphisms.

11. (Even harder) Given a real number $\lambda > 1$ construct a homeomorphism $f : X \to X$ of a compact metric space with $h(f) = \log \lambda$. Hint: First do it for a dense set of Λ's by taking "roots" of full shifts.

12. Entropy can be defined in the same way for a *map* $f : X \to X$ on a compact metric space (not necessarily a homeomorphism). Show that the map $f : S^1 \to S^1$ given by $f(z) = z^2$ has entropy $\log 2$.

    There is a standard way of converting maps to homeomorphisms. Let Σ be the subspace of the infinite product $S^1 \times S^1 \times \cdots$ consisting of sequences $(x_1, x_2, \cdots)$ with $f(x_i) = x_{i-1}$ for $i = 2, 3, \cdots$. This is called the *inverse limit* of the sequence $S^1 \leftarrow S^1 \leftarrow \cdots$ and in this case this space is the *dyadic solenoid*. The map f induces a homeomorphism $F : \Sigma \to \Sigma$ by $F(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. Show that the entropy of F is also $\log 2$.

13. Work through the details of the claim from the class that the 1-sided shift on 2 letters is semi-conjugate to the map $z \mapsto z^2$ on the unit circle in $\mathbb{C}$, and that the entropy of both maps is $\log 2$.

3 Train track maps

14. Start with the automorphism of F_3 (and the map on the rose) given by $a \mapsto Ba$, $b \mapsto AbCb$, $c \mapsto Acc$. First verify that this is not a train track map. Then fold the initial quarter of b with the initial third of c (those that map to A) to improve the situation. Call the edge obtained in this way d, and call b and c the remaining parts of the old b and c. Show that

the new map is given by $a \mapsto BDa$, $b \mapsto dbCb$, $c \mapsto dcdc$, $d \mapsto A$. Note that this graph is not a rose. Also verify that the new map is still not a train track map. Now change the map by a homotopy. The new map is $a \mapsto BDa$, $b \mapsto bCb$, $c \mapsto cdc$, $d \mapsto Ad$. Why is this map homotopic to the previous map? Now verify that this is a train track map by computing the orbit structure on the directions. The PF number will be $2.7166\cdots$.

15. The inverse of the automorphism in Problem 14 is $a \mapsto abacaba$, $b \mapsto abacab$, $c \mapsto abac$. The corresponding map on the rose is a train track map. Show that the dilatation is > 4. Thus automorphisms and their inverses can have different growth rates, unlike (pseudo)-Anosov homeomorphisms.

16. Consider the automorphism of F_n given by $a_1 \mapsto a_2 \mapsto a_3 \mapsto \cdots \mapsto a_n \mapsto a_1a_2$. Prove that the associated map of the rose is a train track map and denote its expansion factor (dilatation) by λ_n. Argue that for $n \geq 3$ $\lambda_n < 1 + \frac{1}{n}$ as follows. Assign lengths to edges by $\ell(a_i) = (1 + \frac{1}{n})^{i-1}$. Then show that the map stretches lengths by $\leq 1 + \frac{1}{n}$ (and in the case of a_n by $< 1 + \frac{1}{n}$). Why does that prove that $\lambda_n < 1 + \frac{1}{n}$? Similarly estimate λ_n from below by $1 + \frac{c}{n}$ for a suitable $c > 0$. Comment: It is known that in every rank there is a smallest possible dilatation > 1 of train track maps in that rank, but the exact value is not known. This is a topic of current research.

17. (Harder) If G is a digraph with n vertices which is oriented-connected and contains a vertex with at least two outgoing edges, prove that the number of oriented paths of length nk is at least 2^k. Deduce that the PF eigenvalue is $\geq \sqrt[n]{2} \geq 1 + \frac{1}{2n}$. Given that train tracks in rank n live on graphs with at most $3n - 3$ edges, deduce that there is a lower bound of the form $1 + \frac{c}{n}$ for some $c > 0$ for all dilatations in rank n that are > 1.

4 Train tracks on surfaces

18. The picture represents the genus 2 surface, where sides are identified in the usual pattern $abABcdCD$ starting at the bottom and going counterclockwise. The puncture is the vertex. The graph is a spine of the surface and a certain homeomorphism induces the following map on the spine:

$$a \mapsto xA, c \mapsto cYcz, x \mapsto ZCu, y \mapsto uaXW, z \mapsto w, w \mapsto XW, u \mapsto ua$$

Show that this is a train track map, compute the (minimal) train track structure, infinitesimal edges, and find the types of singularities the foliations associated to this pseudo-Anosov homeomorphism have.