Notes on the Geometry of Outer space
Preliminary version

Mladen Bestvina

June 19, 2012

0 Introduction

The group $Out(\mathbb{F}_n)$ of outer automorphisms of a free group $\mathbb{F}_n$ of rank n displays many features of mapping class groups and of arithmetic groups. Both of these classes of groups are better understood than $Out(\mathbb{F}_n)$ and they remain to guide the research in $Out(\mathbb{F}_n)$.

More formally, the relationship is as follows. Every outer automorphism $\Phi \in Out(\mathbb{F}_n)$ induces an automorphism of the abelianization $\mathbb{Z}^n$ of $\mathbb{F}_n$, so we have a homomorphism $Out(\mathbb{F}_n) \to GL_n(\mathbb{Z})$, which is always surjective, and it is also injective for $n = 2$. Thus $Out(\mathbb{F}_2) \cong GL_2(\mathbb{Z})$ is very well understood, but for $n > 2$ the group $Out(\mathbb{F}_n)$ still holds many mysteries, in spite of the strides made in the last 30 years.

If S is a closed surface with finitely many punctures (distinguished points) p_i with Euler characteristic $\chi(S-\{p_i\}) < 0$, the mapping class group $MCG(S, \{p_i\})$ is the group of components of the group of homeomorphisms of S that fix the punctures. Equivalently, it is the group of such homeomorphisms modulo isotopy (equivalently, homotopy) fixing the punctures. By passing to π_1 there is a homomorphism $MCG(S, \{p_i\}) \to Out(\pi_1(S-\{p_i\}))$. This homomorphism is always injective, and in the case of the empty set of punctures it is also surjective (the Dehn-Nielsen-Baer theorem). Thus for a nonempty set of punctures we get embeddings of mapping class groups into a suitable $Out(\mathbb{F}_n)$. In the case of the once punctured torus we have an isomorphism, and again $Out(\mathbb{F}_2)$ is well understood from this point of view.

Classically, $Out(\mathbb{F}_n)$ has been studied for as long as mapping class groups, with the works of Nielsen, Magnus, Whitehead and others. Nielsen [16] worked out a finite presentation for $Out(\mathbb{F}_n)$. Magnus [15] studied the kernel of $Out(\mathbb{F}_n) \to GL_n(\mathbb{Z})$ and proved it is finitely generated. It is still open whether this group is finitely presented for $n > 3$ (it isn't when $n = 3$ [13]

and it is trivial when $n = 2$). Whitehead [24, 25] introduced 3-manifold techniques, studying 2-spheres in the n-fold connected sum of $S^1 \times S^2$'s, and gave an algorithm to determine if two elements of $\mathbb{F}_n$ are in the same $Aut(\mathbb{F}_n)$-orbit.

To a topologist, studying free groups is intimately related to studying graphs. Even though Dehn used this point of view very early on (he gave the first proof that subgroups of free groups are free, unfortunately unpublished), the work that followed was dominated by combinatorial group theory and the topological point of view was all but forgotten. Always working with a basis of a free group amounts to restricting oneself to graphs which are roses (wedges of circles).

Stallings [19] reintroduced the graph point of view. He gave an algorithm to construct [the core graph of] the covering space of a graph corresponding to a finitely generated subgroup. The key element of this algorithm is the concept of *folding*, i.e. identifying certain pairs of edges with a common vertex. This operation will play an important role in these notes. Stallings gave very short and elegant proofs of the results about free groups originally proved by the methods of combinatorial group theory.

The watershed event in the study of $Out(\mathbb{F}_n)$ was the introduction of Outer space by Culler and Vogtmann [7]. This event is rightfully referred to as the Big Bang, and it firmly placed topological and geometric methods at the heart of the study of $Out(\mathbb{F}_n)$. Outer space is a contractible space on which $Out(\mathbb{F}_n)$ acts properly discontinuously, and it plays the same role as the symmetric space $SL_n(\mathbb{R})/SO_n$ in the study of $SL_n(\mathbb{Z})$, or as Teichmüller space in the study of mapping class groups. Outer space is primarily a polyhedron and its study involves PL methods, rather than more analytic methods in the case of Teichmüller space. Points in Outer space are graphs equipped with extra structure (metric and marking).

These notes are primarily about the geometry of Outer space. There is a natural metric on Outer space preserved by the action of $Out(\mathbb{F}_n)$. It is defined analogously to Thurston's metric on Teichmüller space [21] as the log of the optimal Lipschitz constant of a map relating two points (i.e. graphs). The metric is not symmetric, but it turns out that this feature is imposed on us by the non-symmetric aspects of $Out(\mathbb{F}_n)$.

There are many exercises throughout the notes. Some are routine, but some require substantial work and stating these exercises was more of an easy alternative for me to writing another section with more details. RATING?

What notes don't cover. Homology, improved train tracks and consequences, structure of R-trees on the boundary, representations, rigidity.

[23]

1 Lecture 1: Outer space and its topology

A *graph* is a cell complex of dimension ≤ 1. The *rose* R_n is the graph with 1 vertex and n edges.

1.0.1 Markings

A *marking* of a graph Γ is a homotopy equivalence $f : R_n \to \Gamma$. This is a convenient way of specifying an identification between $\pi_1(\Gamma)$ with the free group $\mathbb{F}_n$ (thought of as being identified with $\pi_1(R_n)$ once and for all) with a (deliberate) ambiguity of composing with inner automorphisms (no basepoints!). Two marked graphs $f : R_n \to \Gamma$ and $f' : R_n \to \Gamma'$ are *equivalent* if there is a homeomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi f \simeq f'$ (homotopic).

In practice one defines the *inverse* of a marking, i.e. a homotopy equivalence $\Gamma \to R_n$. If the edges of R_n are oriented and labeled by a basis $a, b, \cdots$ of $\mathbb{F}_n$ (thus identifying $\pi_1(R_n) = \mathbb{F}_n$), the inverse marking can be defined by specifying a maximal tree T in Γ, orienting all edges in $\Gamma - T$, and labeling them with a (possibly different) basis of $\mathbb{F}_n$, expressed as words in $a, b, \cdots$. Such a choice defines a map $\Gamma \to R_n$ by collapsing T to a point and sending each edge to the edge path specified by the label.

Exercise 1. Show that the two marked graphs pictured below are equivalent. We follow the convention that capital letters represent inverses of lower case letters. Unlabeled edges form a maximal tree.

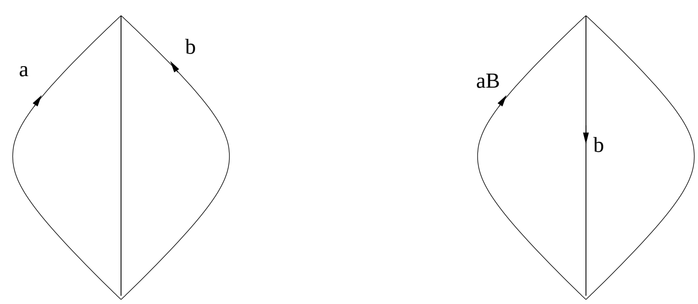

Figure 1: Equivalent marked graphs

1.0.2 Metric

A *metric* on a finite graph is an assignment ℓ of positive numbers $\ell(e)$, called *lengths*, to the edges e of Γ. The *volume* of a finite metric graph is the sum of the lengths of the edges.

A metric on a graph allows one to view the graph as a geodesic metric space, with each edge e having length $\ell(e)$. This point of view lets us assign lengths also to paths in the graph; in particular any closed immersed loop has finite length (an *immersion* is a locally injective map, in this case from the circle to the graph).

We will consider the triples (Γ, ℓ, f) where Γ is a finite graph with all vertices of valence ≥ 3, ℓ is a metric on Γ with volume 1, and $f : R_n \to \Gamma$ is a marking. Two such triples (Γ, ℓ, f) and (Γ', ℓ', f') are *equivalent* if there is an *isometry* (i.e. a length-preserving homeomorphism) $\phi : \Gamma \to \Gamma'$ such that $\phi f \simeq f'$.

Definition 1.1. Outer space

$$\mathcal{X}_n = \{(\Gamma, \ell, f)\}/\sim$$

is the set of equivalence classes of finite marked metric graphs with vertices of valence ≥ 3 and of volume 1.

We will usually omit equivalence class, ℓ and f from the notation, and talk about points $\Gamma \in \mathcal{X}_n$ instead of $[(\Gamma, \ell, f)] \in \mathcal{X}_n$.

1.0.3 Lengths of loops

Once $\pi_1(R_n)$ is identified with $\mathbb{F}_n$ we can view each nontrivial conjugacy class in $\mathbb{F}_n$ as a loop in R_n, up to homotopy. The homotopy class has a unique immersed representative, up to parametrization. If α is a nontrivial conjugacy class and $(\Gamma, \ell, f) \in \mathcal{X}_n$, define the length $\ell_\Gamma(\alpha)$ of α in Γ as the length of the immersed loop homotopic to $f(\alpha)$.

1.1 $\mathbb{F}_n$-trees

If Γ is a marked metric graph, the universal cover $\tilde{\Gamma}$ is a (metric, simplicial) tree, and the marking (i.e. the identification $\pi_1(\Gamma) = \mathbb{F}_n$) induces an action of $\mathbb{F}_n$ on $\tilde{\Gamma}$. The equivalence relation on marked metric graphs translates to saying that two metric simplicial $\mathbb{F}_n$-trees S, T are equivalent if there is an equivariant isometry $S \to T$. Thus $\mathcal{X}_n$ can be alternatively defined as the space of minimal metric simplicial free $\mathbb{F}_n$-trees with covolume 1, up to equivariant isometry. The length of a conjugacy class becomes the translation length in the tree.

The (Gromov) boundary of such a tree, viewed as a 0-hyperbolic graph, is the Cantor set of its ends. If $S, T \in \mathcal{X}_n$ are two trees, any equivariant map

$S \to T$ will be a quasi-isometry and will induce a homeomorphism between the Cantor sets of ends. In this way we may identify all these Cantor sets (with the ambiguity of homeomorphisms induced by the action of $\mathbb{F}_n$ as usual); this is the Cantor set of ends $\partial \mathbb{F}_n$ of $\mathbb{F}_n$.

1.2 Topology and Action

$\mathcal{X}_n$ can be naturally decomposed into open simplices. If Γ is a graph and $f : R_n \to \Gamma$ a marking, the set of possible metrics $M(\Gamma)$ on Γ is an open simplex

$$\{(\ell_1, \ell_2, \cdots, \ell_E) \mid \ell_i > 0, \sum \ell_i = 1\}$$

of dimension $E-1$ if E is the number of edges. If T is a forest (i.e. a disjoint union of trees) in Γ and $\Gamma' = \Gamma/T$ is obtained by collapsing all edges of T to points, then $M(\Gamma')$ can be identified with the open face of $M(\Gamma)$ in which the coordinates of edges in T are 0. Then Γ' is said to be obtained from Γ by *collapsing a forest*, and Γ is obtained from Γ' by *blowing up a forest*. The union $\Sigma(\Gamma)$ of $M(\Gamma)$ with all such open faces as T ranges over all forests in Γ is a simplex-with-missing-faces: it can be obtained from the closed simplex $\{(\ell_1, \ell_2, \cdots, \ell_E) \mid \ell_i \geq 0, \sum \ell_i = 1\}$ by deleting those open faces that assign 0 to edges that do not form a forest. For example, if Γ is the theta-graph with 2 vertices and 3 edges connecting them, $\Sigma(\Gamma)$ is the 2-simplex minus its vertices.

Exercise 2. The smallest dimension of a $\Sigma(\Gamma)$ is n.

Exercise 3. The largest dimension of a $\Sigma(\Gamma)$ is $3n-4$.

Top dimensional simplices correspond to 3-valent graphs, codimension 1 simplices to graphs with one valence 4 vertex and all others valence 3, etc.

The second statement in the following exercise is one of the most difficult in the notes. It establishes that $\mathcal{X}_n$ is a complex of simplices with missing faces (i.e. simplices are determined by their vertices). This fact is not necessary to define simplicial topology and can be omitted on the first reading.

Exercise 4. A face of $\Sigma(\Gamma)$, if nonempty, is of the form $\Sigma(\Gamma')$ for some Γ' obtained from Γ by collapsing a forest. For any Γ, Γ' the intersection $\Sigma(\Gamma) \cap \Sigma(\Gamma')$ is a face of both (or empty).

Hint: The key is to argue that if Γ_1, Γ_2 have a common blowup, then they have a minimal common blowup. This is clear if one restricts to blowups that are all obtained from a fixed blowup Γ by collapsing a forest. In general, the trick is to view all blowups as sitting in a fixed ambient space. For this purpose we use the point of view of trees. Let $T \in \mathcal{X}_n$ be a tree and e an edge of T. Then e determines a partition of the Cantor set of ends $\partial\mathbb{F}_n$ into two compact subsets. The collection of all edges of T determines a collection of partitions with the properties: 1. nesting: Any two sets in these partitions are either nested or disjoint, 2. equivariance: the collection is $\mathbb{F}_n$-equivariant, and 3. finiteness: if $A \subset B$ are two sets in the partitions, there are only finitely many C with $A \subset C \subset B$. Conversely, given a collection of partitions satisfying 1-3 one can build a tree. It will not be free in general, as stabilizers of vertices may be nontrivial, and these trees represent missing faces. (Aside: find an additional property that guarantees that the tree is free.) Now collapsing a forest amounts to passing to a subcollection of partitions. To find the minimal common blowup, just take the union of the collections of partitions. See [9].

In this way $\mathcal{X}_n$ becomes a complex of simplices-with-missing-faces. We define the *simplicial topology* on $\mathcal{X}_n$ just like on a simplicial complex: a subset $U \subset \mathcal{X}_n$ is open [closed] if and only if $U \cap \Sigma(\Gamma)$ is open [closed] in $\Sigma(\Gamma)$ for every $\Gamma \in \mathcal{X}_n$.

Exercise 5. *Reduced Outer space* $\mathcal{R}_n$ is the subspace of $\mathcal{X}_n$ consisting of those graphs that do not have a separating edge. Show that $\mathcal{R}_n$ is an equivariant deformation retract of $\mathcal{X}_n$.

Exercise 6. Identify the link of a simplex $\Sigma(\Gamma)$ with the join of spaces of finite trees with marked points, one for each vertex of valence > 3.

Hint: A vertex of valence m gives the space T_m of finite metric trees of volume 1 with m distinguished points (some of which may coincide) so that every vertex either has valence ≥ 3 or has at least one distinguished point. Thus $T_3 = \emptyset$ and T_4 consists of 3 elements, corresponding to 3 ways of blowing up a valence 4 vertex. It turns out that T_m is homotopy equivalent to the wedge of spheres of dimension $m - 4$. This is a key ingredient in the proof of a theorem of Vogtmann that links in $\mathcal{X}_n$ are homotopy equivalent to wedges of spheres of dimension $3n - 5$. See [22].

{hat}

Exercise 7. There is a simplicial complex $\hat{\mathcal{X}}_n$ and a subcomplex $\hat{\mathcal{X}}_n^{\infty}$ such that $\mathcal{X}_n$ is homeomorphic to $\hat{\mathcal{X}}_n - \hat{\mathcal{X}}_n^{\infty}$. The complex $\hat{\mathcal{X}}_n$ is made up of simplices $\hat{\Sigma}(\Gamma)$ of possibly degenerate metrics on Γ that assign non-negative numbers to the edges with volume 1. The subcomplex $\hat{\mathcal{X}}_n^{\infty}$ consists of marked graphs with degenerate metrics, i.e. those where some nontrivial loop has length 0. $\hat{\mathcal{X}}_n$ is a *flag complex*, i.e. if $v_0, \cdots, v_k$ are vertices such that each pair v_i, v_j spans an edge for $i \neq j$, then the collection spans a simplex.

Exercise 8. Every $\Sigma(\Gamma)$ is contained in only finitely many $\Sigma(\Gamma')$. Conclude that $\mathcal{X}_n$ is locally compact and metrizable.

1.2.1 Thick part and spine

For a fixed small $\epsilon > 0$ define the *thick part* $\mathcal{X}_n(\epsilon)$ of $\mathcal{X}_n$ as the set of $\Gamma \in \mathcal{X}_n$ such that $\ell_\Gamma(\alpha) \geq \epsilon$ for every nontrivial conjugacy class α. When $\epsilon > 0$ is sufficiently small the intersection of $\mathcal{X}_n(\epsilon)$ with every $\Sigma(\Gamma)$ is a nonempty convex set (e.g. taking $\epsilon \leq \frac{1}{3n-3}$ ensures that the barycenter of $\Sigma(\Gamma)$ is in $\mathcal{X}_n(\epsilon)$).

For each simplex-with-missing-faces $\Sigma(\Gamma)$ let $S(\Gamma)$ be the union of simplices in the barycentric subdivision of $\hat{\Sigma}(\Gamma)$ that are contained in $\Sigma(\Gamma)$. Thus $S(\Gamma)$ is the dual of the missing faces. The *spine* $\mathcal{S}_n \subset \mathcal{X}_n$ is the union of $S(\Gamma)$'s for all $\Gamma \in \mathcal{X}_n$.

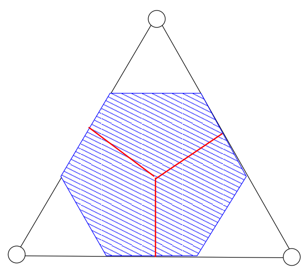

Figure 2: Spine and thick part intersected with a simplex

1.2.2 Action of $Out(\mathbb{F}_n)$

There is a natural right action of $Out(\mathbb{F}_n)$ on $\mathcal{X}_n$ by precomposing the marking. An element $\Phi \in Out(\mathbb{F}_n)$ can be thought of as a homotopy equivalence $\Phi : R_n \to R_n$ and then the action is:

$$(\Gamma, \ell, f) \cdot \Phi = (\Gamma, \ell, f\Phi)$$

The action is simplicial and it is compatible with the action on conjugacy classes:

$$\ell_{\Gamma\Phi}(\alpha) = \ell_\Gamma(\Phi(\alpha))$$

It is sometimes convenient to write $\ell_\Gamma(\alpha)$ as a pairing $\langle \Gamma, \alpha \rangle$ and then the identity becomes

$$\langle \Gamma\Phi, \alpha \rangle = \langle \Gamma, \Phi\alpha \rangle$$

. {finite stabilizers}

Exercise 9. Show that the point stabilizer $Stab(\Gamma, \ell, f)$ is isomorphic to the isometry group $Isom(\Gamma, \ell)$ of the underlying graph, with an isometry ϕ corresponding to the automorphism $f^{-1}\phi f$, where $f^{-1} : \Gamma \to R_n$ denotes the inverse marking.

Exercise 10. Show that there are only finitely many orbits of $\Sigma(\Gamma)$s.

Proposition 1.2. *The action of $Out(\mathbb{F}_n)$ on $\mathcal{X}_n$ is proper. The action on the thick part and on the spine is cocompact.*

Exercise 11. (Combinatorial description of the spine.) Show that the following simplicial complex $\mathcal{P}$, the *poset of marked graphs* is homeomorphic to the spine $\mathcal{S}$. The vertices of $\mathcal{P}$ are marked graphs (Γ, f) (with Γ having no vertces of valence ≤ 2) modulo equivalence $(\Gamma, f) \sim (\Gamma', f')$ if there is a homeomorphism $\phi : \Gamma \to \Gamma'$ with $\phi f \simeq f'$. A k-simplex in $\mathcal{P}$ is induced by a sequence of nontrivial forest collapses $\Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_k$.

Exercise 12. $\dim \mathcal{S} = 2n - 3$.

Exercise 13. There are equivariant deformation retractions from Outer space $\mathcal{X}_n$ to the thick part $\mathcal{X}_n(\epsilon)$ (for small $\epsilon > 0$) and from $\mathcal{X}_n(\epsilon)$ to the spine $\mathcal{S}$.

1.3 Rank 2 picture

Since $Out(F_2) \cong GL_2(\mathbb{Z}) \cong MCG(T^2, \{p\})$, the symmetric space $SL_2(\mathbb{R})/SO_2$ and Teichmüller space of $(T^2, \{p\})$ is hyperbolic plane $\mathbb{H}^2$, it is not surprising that $\mathcal{X}_2$ is essentially also (a combinatorial version of) $\mathbb{H}^2$. More precisely, the reduced Outer space in rank 2 is the *filled in Farey graph* pictured in Figure 3.

Markings of three of the simplices are pictured in Figure 4. Observe that there are two ways to blow up a rose R_2 to a theta graph and this translates into the fact that reduced Outer space is a surface.

To obtain the whole Outer space, we also need to attach simplices corresponding to graphs with seprating edges, see Figure 5. These simplices have missing vertices and two of the sides, and are attached to the reduced Outer space along the third side.

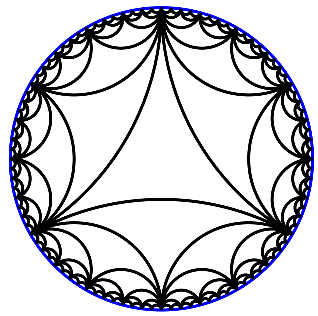

Figure 3: Reduced Outer space in rank 2. The circle and the vertices of the triangles are not part of the space. {farey}

1.4 Contractibility

The central fact about Outer space is its contractibility, proved by Marc Culler and Karen Vogtmann.

Theorem 1.3. *[7] Outer space $\mathcal{X}_n$ is contractible.*

Of course, this means that the thick part and the spine are also contractible.

Culler-Vogtmann use combinatorial Morse theory and argue that the spine is contractible. They carefully order the set of roses in $\mathcal{X}_n$: $r_1, r_2, \cdots$ and argue that for each i the union of stars of the first i roses is contractible. The difficult step is showing that the intersection of the star of the ith rose with the union of the previous stars is contractible.

An alternative proof, more in the spirit of these notes, was constructed by Skora [18], building on the ideas of Steiner [20]. For each $\Gamma \in \mathcal{X}_n$ they construct a (folding) path from a point in the simplex containing the rose with identity marking to R_n and argue that the collection of these paths varies continuously in Γ (this is technically the hard step). These paths then determine a deformation retraction from $\mathcal{X}_n$ to a simplex with missing faces. For more on folding paths see Lecture 2.

Neither Steiner's nor Skora's work was published; for details see [5].

Corollary 1.4. *$Out(\mathbb{F}_n)$ is finitely presented.*

Proof. Recall that if a group acts freely and cocompactly on a simply connected simplicial complex, then it is finitely presented. More generally, it is finitely presented if it acts cocompactly on a simply connected complex with

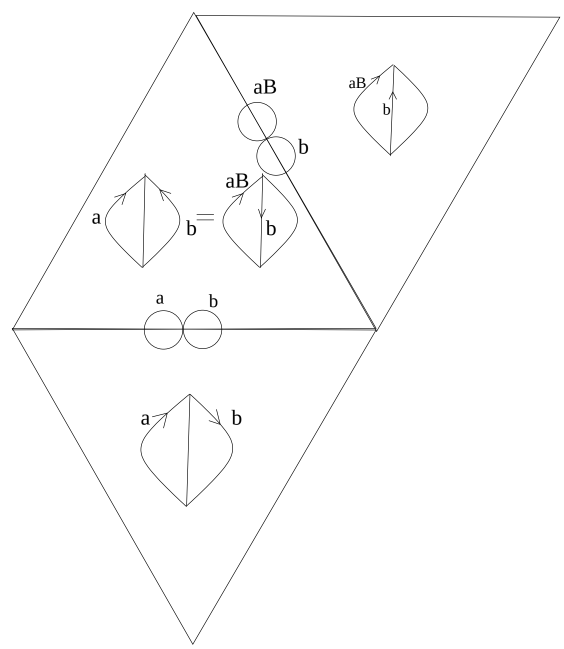

Figure 4: 3 2-simplices with their marked graphs {markings}

finitely presented vertex stabilizers and finitely generated edge stabilizers. The action on the spine has finite stabilizers. □

Proposition 1.5. *$Out(\mathbb{F}_n)$ is virtually torsion-free.*

Proof. We claim that the kernel of $Out(\mathbb{F}_n) \to GL_n(\mathbb{Z}/3)$ is torsion-free. Let $1 \neq \Phi \in Out(\mathbb{F}_n)$ have finite order. By Exercise 48 (see also Exercise 49) Φ is realized as a graph isomorphism $\phi : \Gamma \to \Gamma$. We may collapse all separating edges of Γ, so every edge is contained in an embedded circle. If Φ is in the kernel, then ϕ maps any circle to itself preserving orientation. But for each circle C in Γ there is another circle C' so that $C \cap C'$ is nonempty, connected, and $\neq C$. Thus ϕ is identity. □

Corollary 1.6. *A torsion-free subgroup H of finite index has a compact classifying space $K(H, 1)$ of dimension $2n-3$, and the virtual cohomological dimension of $Out(\mathbb{F}_n)$ is $2n-3$.*

To see that $vcd(Out(\mathbb{F}_n)) \geq 2n-3$ note that $Out(\mathbb{F}_n)$ contains an abelian subgroup of rank $2n-3$. E.g. for $n = 3$ we can take the group of automorphisms of the form $a \mapsto a$, $b \mapsto a^p b$, $c \mapsto a^q c a^r$ for $p, q, r \in \mathbb{Z}$.

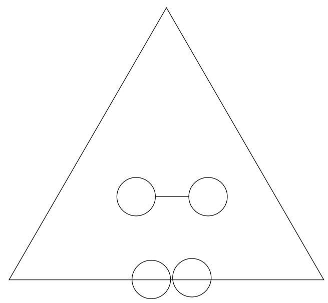

Figure 5: Simplices corresponding to graphs with separating edges {separating}

Exercise 14. Find a nontrivial element of finite order in the kernel of $Out(\mathbb{F}_n) \to GL_n(\mathbb{Z}/2)$. Show that every such element has order 2 and that therefore every finite subgroup of the kernel is abelian (in fact, a direct sum of $\mathbb{Z}/2$'s). Can you find the largest such subgroup?

Exercise 15. Can you find estimates on the size of the largest finite subgroup of $Out(\mathbb{F}_n)$? For example, the stabilizer of a rose has order $2^n n!$. Can you find a larger finite group? What about $n = 2$ and 3? What is the largest symmetric group contained in $Out(\mathbb{F}_n)$?

1.5 The sphere complex

There is an alternative point of view on Outer space, introduced by Hatcher [12], in the spirit of Whitehead [24, 25]. It is easiest to give Hatcher's description the complex $\hat{\mathcal{X}}_n$ from Exercise 7. Let W be the connected sum of n copies of $S^1 \times S^2$. A vertex of $\hat{\mathcal{X}}_n$ is an isotopy class of essential 2-spheres (i.e. those that don't bound a 3-ball). A collection of these span a simplex if they can be realized disjointly (in their isotopy classes). Of course, a point in such a simplex is specified by assigning barycentric coordinates (weights) to the spheres, and taking the dual graph with lengths of edges equal to these coordinates gives a correspondence between the sphere model and the graph model of $\mathcal{X}_n$. To argument that this construction describes the same complex, as well as the relationship between the mapping class group of W and $Out(\mathbb{F}_n)$, uses some classical 3-manifold topology, particularly the work of Laudenbach [14]. Hatcher also gives a proof that $\mathcal{X}_n$ is contractible from this point of view. He constructs *surgery paths* between two weighted collections of spheres and argues continuity, as in the Steiner-Skora approach. The main technical tool in Hatcher's work is the notion of normal form of one collection of disjoint spheres with respect to another, minimizing the number of components of the intersection.

1.6 Other definitions of topology

The *length function* is the function

$$\mathcal{L} : \mathcal{X}_n \to (0, \infty)^{\mathcal{C}}$$

that to Γ assigns $(\alpha \mapsto \ell_\Gamma(\alpha))$. We will see in Lecture 2 that this function is injective, and if we identify $\mathcal{X}_n$ with the image, the subspace topology induces a topology on $\mathcal{X}_n$, the *length function topology.* In exercises in Lecture 2 it is shown that this topology is equivalent to the simplicial topology, i.e. $\mathcal{L}$ is an embedding.

In the same way, after projectivizing (passing to the quotient under scaling), we have a map $\mathcal{X}_n \to \mathbb{P}((0, \infty)^{\mathcal{C}})$. This function is also injective and the induced topology, the *projective length function topology*, is equivalent to the simplicial topology (see Lecture 2).

There are two more topologies induced by the Lipscitz metric, and they are both equivalent to the simplicial topology. See Lecture 2.

Finally, from the point of view of trees, there is the equivariant Gromov-Hausdorff topology, see [17]. It is also equivalent to the simplicial topology.

1.7 Compactification

Proposition 1.7 ([6])**.** *The image of $\mathcal{X}_n$ in $\mathbb{P}([0, \infty)^{\mathcal{C}}) = [0, \infty)^{\mathcal{C}} - \{0\}/\mathbb{R}_+$ has compact closure.*

This can be proved as follows.

Exercise 16. Fix $f : \mathcal{C} \to (0, \infty)$ and $\alpha \in \mathcal{C}$. The subset $K_{f,\alpha}$ of $\mathbb{P}([0, \infty)^{\mathcal{C}})$ consisting of $g : \mathcal{C} \to (0, \infty)$ with $g(\alpha)/f(\alpha) \geq g(\beta)/f(\beta)$ for all β has compact closure. {Kf}

It will follow from Lecture 2 that the image of $\mathcal{X}_n$ is covered by finitely many sets of the form $K_{f,\alpha}$. See Exercise 24.

The compactification $\overline{\mathcal{X}_n}$ is the closure of $\mathcal{X}_n \subset P([0, \infty)^{\mathcal{C}})$. The group $Out(\mathbb{F}_n)$ continues to act on the closure. The ideal points can be interpreted as $\mathbb{R}$-trees with an isometric action of $\mathbb{F}_n$, but these actions in general are neither free nor simplicial, and in fact typically the set of branch points is dense. It is beyond the scope of these lectures to discuss $\mathbb{R}$-trees. For an introduction, see e.g. [2]. We will restrict ourselves to a few examples below. Formally, an $\mathbb{R}$-tree is a geodesic metric space which is 0-hyperbolic, or equivalently, the unique segment joining two distinct points is isometric to a closed interval in $\mathbb{R}$.

Exercise 17. The closure of each simplex-with-missing-faces $\Sigma(\Gamma)$ is the closed simplex of possibly degenerate metrics.

For example, the degenerate metric on R_2 that assigns length 1 to one edge and length 0 to the other can be interpreted as the quotient, as in Bass-Serre theory, of an isometric action of F_2 on a simplicial tree corresponding to the splitting $F_2 = \mathbb{Z} *_1$.

For a more typical example, start with a foliation of $\mathbb{R}^2$ by parallel lines with irrational slope. The leaf space of this foliation is $\mathbb{R}$. Fix a translation invariant metric on the leaf space; this is a transverse measure on the foliation. Consider the induced foliation of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ and remove a point $p \in T$. The fundamental group of $T - \{p\}$ is F_2 and it acts on the universal cover U preserving the preimage foliation. Then the leaf space L of this foliation is an $\mathbb{R}$-tree and F_2 acts on it isometrically. The distance between two leaves can be defined as the infimum of measures of PL paths joining the two leaves; a straigh line segment in $T - \{p\}$ or its universal cover has a natural measure given by the transverse measure, i.e. by taking the length of the projection in the leaf space of the foliation of $\mathbb{R}^2$. There are countably many leaves in $\widetilde{T - \{p\}}$ with pairwise distance 0; they correspond to lifts of rays starting at p and related by short paths going around p with argument a multiple of π. All these are identified to one point in L.

Alternatively, let Λ be a measured geodesic lamination on a hyperbolic surface S with cusps. Then the leaf space of the universal cover is an $\mathbb{R}$-tree with an isometric action of $\pi_1(S)$, and this action represents an ideal point in the boundary of Outer space of $\pi_1(S)$.

We'll see later examples of more interesting $\mathbb{R}$-trees, e.g. stable trees of automorphisms.

Exercise 18. Show that the image of $\mathcal{L} : \mathcal{X}_n \to [0,\infty)^{\mathcal{C}}$ does not have compact closure. Are there divergent sequences in $\mathcal{X}_n$ whose images by $\mathcal{L}$ converge?

1.7.1 Compactification in rank 2

We now describe the compactification in rank 2. For more details see [8]. Since $Out(F_2) \cong GL_2(\mathbb{Z})$ and reduced Outer space is $\mathbb{H}^2$ one might expect that the compactified reduced Outer space is a closed disk. This is indeed the case, however, the set of ideal points is not just a circle, but a circle with countably many arcs attached to a dense set of points on the circle, with arcs forming a null-sequence (i.e. the diameters go to 0). See Figure 6.

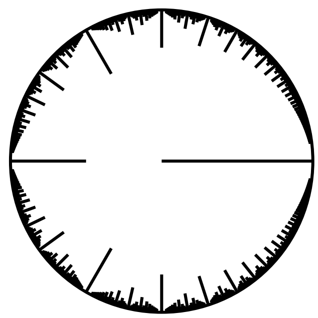

Figure 6: The compactification of reduced Outer space in rank 2 is a 2-disk with the set of ideal points forming a circle with infinitely many arcs attached {circle and arcs}

Points on the circle where no arc is attached ("irrational points") are represented by $\mathbb{R}$-trees dual to irrational slope foliations, as described above. All conjugacy classes have nonzero translation length except for the powers of the commutator $[a, b]$ of the basis elements (the conjugacy class of the commutator is preserved by every automorphism; there is no nontrivial conjugacy class preserved by every automorphism in higher rank, in fact, there are individual automorphisms that don't preserve any nontrivial conjugacy class).

Figure 7 gives a fragment of the Farey triangulation and how it fits with the arcs in the compactification. Consider the rose R_2 with identity marking and the automorphism Φ that sends $a \mapsto a$, $b \mapsto ab$. This automorphism fixes a (missing) vertex obtained from R_2 by collapsing a to a point. This vertex is one endpoint of an arc in the compactification; the other endpoint is attached to the circle.

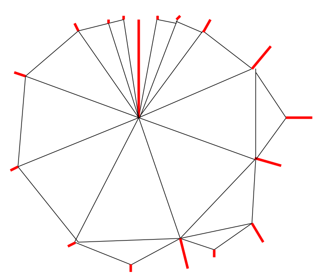

Figure 7: A fragment of the triangulation and convergence to arcs. {dehn}

Every point on the arc is represented by a simplicial tree, see Figure 8.

When the arc labeled a degenerates to a point, the picture represents an endpoint of the arc (a missing vertex). When it degenerates to the whole circle, it represents the other endpoint, the one attached to the circle. This simplicial tree is also the dual to a simple closed curve on a punctured torus.

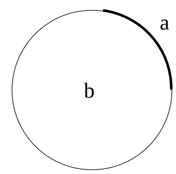

Figure 8: The Bass-Serre quotient of the simplicial tree representing a point in the interior of an arc. {arc}

Exercise 19. Let Γ_i be the rose labeled a and $a^{-i}b$ with both edges of length $\frac{1}{2}$ and let Φ be the automorphism $a \mapsto a$, $b \mapsto ab$.

- Show that $\Gamma_i = \Gamma_0 \Phi^i$.
- Show that the sequence Γ_i converges to the endpoint of an arc that's on the circle. Sketch a possible location of Γ_i in Figure 7.
- Modify the metric of each Γ_i so the sequence converges to an arbitrary point on this arc.

Potential additional topics: explicit finite generating set via wh moves, characterization of trees in the boundary.

2 Lecture 2: Lipschitz metric, Train tracks

In this Lecture we introduce the Lipschitz metric on Outer space. The definition dates back to the 1990's when my former student Tad White proved the key Lemma 2.3. At the time we didn't have any applications for this metric. It recaptured my own interest when I realized that one can give a classification of automorphisms in the style of Bers using this metric. Bers [?] proved the Thurston classification theorem for mapping classes using the Teichmller metric on Teichmüller space. This is the subject of Lecture 3.

Francaviglia and Martino were the first to study this metric sytematically. Much of the material in this section is in their paper [10].

2.1 Definitions

Let $[(\Gamma, \ell, f)], [(\Gamma', \ell', f')]\mathcal{X}_n$ be two points in Outer space. A map $\phi : \Gamma \to \Gamma'$ is a *difference of markings* map if $\phi f \simeq f'$. We will only consider Lipschitz maps and we denote by $\sigma(\phi)$ the Lipschitz constant of ϕ. When ϕ is homotoped rel vertices to a map ϕ' which has constant slope on each edge, then $\sigma(\phi') \leq \sigma(\phi)$. We define the distance:

$$d(\Gamma, \Gamma') = \inf_{\phi} \log \sigma(\phi)$$

as $\phi : \Gamma \to \Gamma'$ ranges over all difference of markings. Recall the Arzela-Ascoli theorem, which says that any sequence of L-Lipschitz maps between two compact metric spaces has a convergent subsequence. This theorem implies that infimum above is realized. We will call a difference of markings $\phi : \Gamma \to \Gamma'$ *optimal* if it has constant slope on each edge and minimizes the Lipschitz constant (which is then the maximal slope).

2.2 Elementary facts

Proposition 2.1.

- *$d(\Gamma_1, \Gamma_3) \leq d(\Gamma_1, \Gamma_2) + d(\Gamma_2, \Gamma_3)$.*
- *$d(\Gamma, \Gamma') \geq 0$ and equality implies $\Gamma = \Gamma'$.*
- *$d(\Gamma\Phi, \Gamma'\Phi) = d(\Gamma, \Gamma')$.*

Proof. The first claim follows from the general fact that $\sigma(\psi\phi) \leq \sigma(\psi)\sigma(\phi)$. For the second claim, let $\phi : \Gamma \to \Gamma'$ be an optimal map. If $d(\Gamma, \Gamma') < 0$ then all slopes of ϕ are < 1. This implies that the volume of the image of ϕ is < 1, so ϕ is not surjective. But a homotopy equivalence between finite graphs without vertices of valence 1 is always surjective.

If $d(\Gamma, \Gamma') = 0$ then all slopes of ϕ must be equal to 1 and the images of different edges can intersect only in finite sets. Thus ϕ is a quotient map that identifies finitely many collections of finitely many points. The only way for such a map to be a homotopy equivalence (or even for Γ and Γ' to have the same rank) is for ϕ to be an isometry, so $\Gamma = \Gamma'$.

The third claim is an exercise. □

2.3 Example {ss:ex}

To illustrate the definition, let us compute the distance in the following example, see Figure 9.

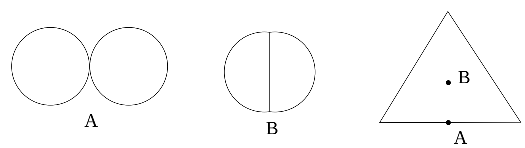

Figure 9: A is the rose with edge lengths $\frac{1}{2}$ and B is the theta graph with edge lengths $\frac{1}{3}$, bot in the same 2-simplex. {distance}

To compute $d(A, B)$ consider the difference of markings map ϕ that sends the vertex to the midpoint of the middle edge, the loop on the left homeomorphically to the circle formed by the middle and the left edges, and the loop on the right homeomorphically to the circle formed by the middle and the right edge. The slope of ϕ on both edges is $\frac{(\frac{2}{3})}{(\frac{1}{2})} = \frac{4}{3}$, so $d(A, B) \leq \log \frac{4}{3}$. We now claim that $d(A, B) = \log \frac{4}{3}$. To see this, observe that each of the two edges in A is a loop of length $\frac{1}{2}$ and any difference of markings map will map it to a loop homotopic to an immersed loop of length $\frac{2}{3}$. Thus the length of the image cannot be smaller than $\frac{2}{3}$, and so the slope of any difference of markings map on either edge cannot be less than $\frac{4}{3}$. More generally, we observe:

Lemma 2.2. *If α is any nontrivial conjugacy class then*

$$\log \frac{\ell_{\Gamma'}(\alpha)}{\ell_{\Gamma}(\alpha)} \leq d(\Gamma, \Gamma')$$

So for any α we obtain a lower bound to the distance. In our example, the lower bound agrees with the upper bound provided by the explicit difference of markings map. This determines the distance.

We will say that a conjugacy class is a *witness* if equality holds in the statement of the Lemma.

In a similar way, one can compute that $d(B, A) = \log \frac{3}{2}$ by considering the map $B \to A$ that collapses the middle edge, and the witness loop formed by the other two edges.

Note in particular that $d(A, B) \neq d(B, A)$.

Exercise 20. Let A be as above, and let C_ϵ be the graph in the same 1-simplex as A with lengths of edges ϵ and $1 - \epsilon$. Show that $d(C_\epsilon, A) \to \infty$ as $\epsilon \to 0$, but $d(A, C_\epsilon)$ stays bounded by $\log 2$.

Thus the distance function is not even quasi-symmetric, i.e. $\frac{d(X,Y)}{d(Y,X)}$ can be arbitrarily large. However, a theorem of Handel-Mosher [11] states that the restriction of d to any thick part $\mathcal{X}_n(\epsilon)$ *is* quasi-symmetric. More generally, by [1], there is a "potential function" $\Psi : \mathcal{X}_n \to [0, \infty)$ which is $Out(\mathbb{F}_n)$-invariant and such that

$$d(\Gamma', \Gamma) \leq K d(\Gamma, \Gamma') + M(\Psi(\Gamma') - \Psi(\Gamma))$$

for universal constants $K, M > 0$. By the continuity of Ψ the second term is bounded on any thick part.

Exercise 21. Optimal maps between two graphs are not unique. Show that the set of optimal maps $A \to B$ in the example above forms a closed interval. Show that in general the set of optimal maps is naturally a compact convex polytope.

2.4 Tension graph, train track structure

Here is the crucial fact. It is analogous to Teichmüller's theorem for Riemann surfaces. It states that witnesses always exist.

{white}

Lemma 2.3. *Suppose* $d(\Gamma, \Gamma') = \log \lambda$. *Then there is a conjugacy class* $\alpha \in \mathcal{C}$ *such that*

$$\frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)} = \lambda$$

Note that for *any* α inequality $\leq$ holds. So the lemma says that we can define the distance alternatively as

$$d(\Gamma, \Gamma') = \log \max_\alpha \frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)}$$

The equality between the min and the max is an instance of the *max-flow min-cut* principle.

The proof introduces the key idea of train tracks.

Proof. Fix a difference of markings map $\phi : \Gamma \to \Gamma'$ with $\sigma(\phi) = \lambda$. By $\Delta = \Delta_\phi$ denote the union of edges of Γ on which the slope of f is λ. This subgraph of Γ is called the *tension graph* for ϕ, and it may have vertices of valence 1 or 2. Now let v be a vertex of Δ. A *direction* at v in Δ is a germ of geodesic paths $[0, \epsilon] \to \Delta$ sending 0 to v. Alternatively, it is an oriented edge of Δ with initial vertex at v. Denote the set of these directions by $T_v(\Delta)$. Its cardinality is the valence of v in Δ and this set plays the role of the unit tangent space at v. Now ϕ induces a map (kind of a derivative)

$$phi_* : T_v(\Delta) \to T_{\phi(v)}(\Gamma')$$

since it sends a geodesic $\gamma : [0, \epsilon] \to \Delta$ to a geodesic $\phi\gamma : [0, \epsilon] \to \Gamma'$ (parametrized with speed λ). Here $\phi(v)$ may not be a vertex, in which case $T_{\phi(v)}(\Gamma')$ naturally has two directions. Thus we have an equivalence relation on $T_v(\Delta)$:

$$d_1 \sim d_2 \iff \phi_*(d_1) = \phi_*(d_2)$$

A *train track structure* on a graph Δ is simply a collection of equivalence relations on the sets $T_v(\Delta)$ for all vertices v. Thus the tension graph is naturally equipped with a tran track structure.

It is customary to draw equivalent directions as tangent to each other. The equivalence classes are *gates*. An immersed path in Δ (thought of as a train route) is *legal* if whenever it passes through a vertex, the entering and the exiting gates are distinct. Otherwise, a path is *illegal*. Similarly, a turn (i.e. a pair of distinct directions) is illegal if the directions are equivalent; otherwise the turn is legal. More informally, legal paths do not make 180° turns.

Figure 10 shows the tension graphs with their train track structures from the examples in section 2.3. The tension graph of $\phi : A \to B$ is all of A and the vertex has two gates. For the map $B \to A$ the tension graph is a circle formed by two edges and all turns are legal.

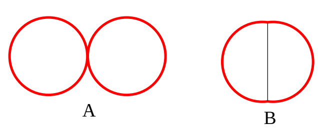

Figure 10: Tension graphs with their train track structures from examples in 2.3. {tt}

Now we make the following two observations:

- if the immersed loop $\alpha|\Gamma$ representing conjugacy class α in Γ is contained in Δ and is legal, then $\frac{\ell_{\Gamma'}(\alpha)}{\ell_\Gamma(\alpha)} = \lambda$,

- if every vertex of Δ has at least two gates, then Δ contains a legal loop; in fact this loop can be chosen to cross every oriented edge at most once.

The first of these claims is an exercise in definitions: f has slope λ on each edge of $\alpha|\Gamma$ and consecutive edges are mapped without backtracking by definition of legality. For the second claim, keep extending a legal path until the same oriented edge repeats.

Of course, in general Δ may have vertices with one gate. To finish the proof we will show that ϕ may be perturbed so that every vertex has at least two gates.

Claim: Suppose v is a vertex of Δ_f with only one gate. Then ϕ may be perturbed to $\phi' : \Gamma \to \Gamma'$ so that $\sigma(\phi') = \lambda$ and $\Delta_{\phi'} \subsetneq \Delta_\phi$.

Repeating this operation will eventually produce a perturbation of ϕ whose tension graph has at least two gates at every vertex (note that the set of edges where the slope is λ cannot become empty by the assumption that $d(\Gamma, \Gamma') = \log \lambda$).

Proof of Claim. The homotopy ϕ_t from ϕ to ϕ' will be stationary on all vertices except for v, and it will move $\phi(v)$ slightly in the direction $\phi_*(d)$, where $d \in T_v(\Delta)$. All maps ϕ_t are linear on edges. Thus the slope is unaffected on edges not incident to v, it decreases on edges in Δ incident to v, and it may increase on edges outside Δ incident to v. The perturbation is small so that even the increased slope on such edges is $< \lambda$. Thus $\Delta_{\phi'} \subset \Delta_\phi$ but $\Delta_{\phi'}$ does not contain v and edges incident to it. □

Exercise 22. [10] Show that in any graph with a train track structure with at least two gates at every vertex, there is a legal loop that is either embedded, or it forms a "figure 8" crossing each edge once, or it forms a "dumbbell", crossing edges in the two loops once and edges in the connecting arc twice. See Figure 11. {candidates}

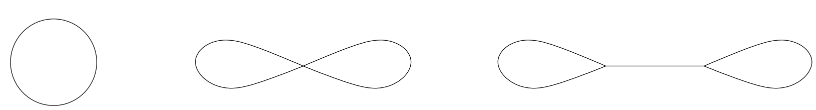

Figure 11: Possible forms of candidates. Train track structure is suggested by the pictures. {p:candidates}

We say that an immersed loop in a graph Γ is a *candidate* if it has a form as in Exercise 22. Thus there is always a candidate which is a witness and there is a simple algorithm to compute distances $d(\Gamma, \Gamma')$ in Outer space.

Simply look at the ratio of lengths in Γ' and in Γ of all candidate loops in Γ and take the log of the largest such ratio.

{find distance}

Exercise 23. Let R be the rose in $\mathcal{X}_3$ with all edges of length $\frac{1}{3}$ and with inverse marking given by a, b, c, and let Γ be another such rose but with inverse marking given by $acA, bacB, a$. Find all candidates in each that are witnesses for the distance to the other.

{Kf2}

Exercise 24. Show that the image of $\mathcal{X}_n \to \mathbb{P}([0,\infty)^{\mathcal{C}})$ is covered by finitely many sets of the form $K_{f,\alpha}$ (see Exercise 16). This finishes the proof that the closure of the image is compact.

Exercise 25. Consider the automorphism Φ of $\mathbb{F}_{n4} = \langle a, b, c, d\rangle$ given by $a \to b \to c \to d \to ADCB$ (capital letters are inverses of the lowercase letters).

(a) Let R be the rose with the identity marking (so the edges correspond to a, b, c, d) and with all lengths $\frac{1}{4}$. Compute $d(R, R\Phi)$.

(b) Find the graph Γ in the same simplex as R (i.e. the same marking, but edge lengths can be arbitrary) so that $d(\Gamma, \Gamma\Phi)$ is minimal.

(c) Can you find a graph Γ' in a small neighborhood of Γ so that $d(\Gamma', \Gamma'\Phi) < d(\Gamma, \Gamma\Phi)$?

Exercise 26. If Γ, Γ' are distinct points in $\mathcal{X}_n$ show that there are conjugacy classes α, β such that $\ell_\Gamma(\alpha) > \ell'_\Gamma(\alpha)$ and $\ell_\Gamma(\beta) < \ell'_\Gamma(\beta)$. Deduce that the length function $:\mathcal{X}_n \to (0,\infty)^{\mathcal{C}}$ and the projectivized length function $\mathcal{X}_n \to (0,\infty)^{\mathcal{C}}$ are injective.

{candidates+}

Exercise 27. For a marked graph Γ let K_Γ be the finite set of candidates for Γ and for all marked graphs obtained from Γ by collapsing a forest. Show that lengths of elements of K_Γ determine each point of $\Sigma(\Gamma)$.

The following two properies of the Lipschitz metric point out similarities with the ℓ^∞ metric.

{straight}

Exercise 28. In each simplex straight lines are geodesics (not necessarily parametrized with unit speed).

Hint: Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three points along a straight line with Γ_2 between the other two. Argue that any witness for $\Gamma_1 \to \Gamma_3$ is also a witness for $\Gamma_1 \to \Gamma_2$ and for $\Gamma_2 \to \Gamma_3$.

Exercise 29. Show that geodesics are not unique in general. Specifically, in rank 2, show that there are geodesics contained in a 2-simplex with endpoints on one edge, but with the geodesic intersecting the interior.

Exercise 30. The distance function $d : \mathcal{X}_n \times \mathcal{X}_n \to [0, \infty)$ is continuous.

Hint: It suffices to prove continuity on each simplex. But on a simplex the distance is determined by lengths of finitely many conjugacy classes, see Exercise 27.

One can define a topology on $\mathcal{X}_n$ using the distance function. For $\Gamma \in \mathcal{X}_n$ and $R > 0$ define the *forward open ball*

$$B_{\to}(\Gamma, R) = \{\Gamma' \in \mathcal{X}_n \mid d(\Gamma, \Gamma') < R\}$$

and the *backward open ball*

$$B_{\leftarrow}(\Gamma, R) = \{\Gamma' \in \mathcal{X}_n \mid d(\Gamma', \Gamma) < R\}$$

In a similar way define closed balls, replacing the strict inequality with $\leq$. The collection of all forward [backward] open balls defines a topology on $\mathcal{X}_n$ (but even that is not obvious due to asymmetry of the metric). Exercises below outline a proof that both of these topologies are equivalent to the length function topology, first on a simplex, and then in Section 2.5 on all of $\mathcal{X}_n$.

Exercise 31. Show that for every $\Delta \in \Sigma(\Gamma)$ and any $R > 0$ the intersection of $\Sigma(\Gamma)$ with a closed forward [backward] ball centered at Δ is a polytope, i.e. it is the intersection of finitely many halfspaces, and its interior (with respect to the standard topology on the simplex) is the intersection with the corresponding open ball.

Exercise 32. Show in addition that intersections of $\Sigma(\Gamma)$ with closed backward balls are compact, while the intersectons with closed forward balls may not be compact.

Exercise 33. Show that on each $\Sigma(\Gamma)$ open forward [backward] balls define the standard topology.

Hint: Each open ball is open in the standard topology. Use Exercise 28 and continuity of the metric to argue that balls of small radius are contained in prechosen neighborhoods of the center.

2.5 Folding paths

{folding paths}

A folding path is determined by an optimal map $\phi : \Gamma \to \Gamma'$ such that the tension graph Δ_ϕ is all of Γ and every vertex has at least two gates. It is a geodesic path Γ_t from Γ to Γ' and for each $t < t'$ it comes with an optimal map $\Gamma_t \to \Gamma_{t'}$ so that the tension graph is all Γ_t and these maps compose correctly for $t < t' < t''$. To define an initial segment of this path choose a small $\epsilon > 0$ and for $t \in [0, \epsilon]$ define Γ_t by identifying segments of length t issuing from any vertex in equivalent directions. Then rescale to make volume equal to 1.

For example, for the map $\phi : A \to B$ considered in section 2.3 time t graph before rescaling would have one edge of length $2t$ and two edges of length $1 - 2t$.

There are naturally induced maps $\Gamma_t \to B$ so at $t = \epsilon$ one can repeat the procedure to continue the path.

It is not clear *a priori* that this defines a path globally. If ϕ is simplicial with respect to some subdivisions of Γ and Γ' and the lengths of all edges in each subdivision are equal, the procedure amounts to Stallings' folding, identifying to edges whenever they share a vertex and map to the same edge (but here we do it continuously resulting in a path in $\mathcal{X}_n$.

A very elegant definition of folding paths is due to Skora [18]. It is most conveniently described in terms of the universal cover $\tilde{\phi} : \tilde{\Gamma} \to \Gamma'$. Consider the graph of $\tilde{\phi}$:

$$Gr(\tilde{\phi}) = \{(u, v) \in \tilde{\Gamma} \times \tilde{\Gamma}' \mid \tilde{\phi}(u) = v\}$$

and define the *vertical t-neighborhood* of the graph $Gr(\tilde{\phi})$:

$$N_t = \{(u, v) \in \tilde{\Gamma} \times \tilde{\Gamma}' \mid d(\tilde{\phi}(u), v \leq t\}$$

where d refers to the path metric on $\tilde{\Gamma}'$. Restrict the foliation of $\tilde{\Gamma} \times \tilde{\Gamma}'$ by $\{u\} \times \tilde{\Gamma}'$, $u \in \tilde{\Gamma}'$ to N_t and define $\tilde{\Gamma}_t$ as the quotient space where all components of leaves are collapsed. Then $\tilde{\Gamma}_t$ is a tree and its quotient by the action of $\mathbb{F}_n$ is the desired graph Γ_t (which needs to be rescaled). For $t = 0$ we have $\Gamma_t = \Gamma$ and for t large $\Gamma_t = \Gamma'$.

To get a feel for this definition, consider the "tent map" $\phi : [-1, 1] \to [0, \lambda]$ for $\lambda > 0$, which has slope λ on $[-1, 0]$ and slope $-\lambda$ on $[0, 1]$. The graph of this map is pictured in Figure 12 (with the target thought of as $\mathbb{R}$).

The metric on $\tilde{\Gamma}_t$ comes from projecting to the first coordinate and maps $\Gamma_t \to \Gamma_{t'}$ for $t < t'$ from inclusion $N_t \hookrightarrow N_{t'}$.

To see that a folding path is alwas a geodesic take any legal loop in Γ and observe that its image in Γ_t is legal for $\Gamma_t \to \Gamma_{t'}$ for any $t' > t$ and that it is a witness for that map (the slope of the map on each edge is the ratio of lengths of the loop at $\Gamma_{t'}$ and Γ_t.

Example 2.4. Let Γ be the rose in $\mathcal{X}_2$ with identity marking, and with $\ell(a) = \lambda^{-2}$ and $\ell(b) = \lambda^{-1}$ where $\lambda > 0$ satisfies $\lambda^{-1} + \lambda^{-2} = 1$ (see Example 3.5). Let $\phi : \Gamma \to \Gamma\Phi$ be the optimal map for Φ given by $a \mapsto b$, $b \mapsto ab$ suggested by Φ, so ϕ has slope λ on both edges and $\Delta = \Gamma$. The

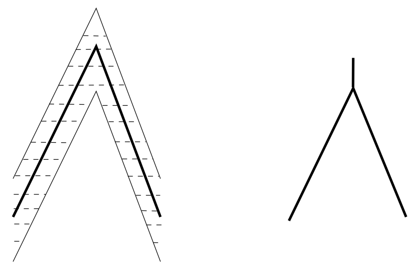

Figure 12: Construction of folding paths following Skora. {skora}

folding path fromΓ to $\Gamma\Phi$ amounts to identifying the terminal portion of the edge b around the edge a in the direction of A (note that $\{A, B\}$ is the only illegal turn).

Proposition 2.5. *[10] d is a geodesic metric.* {geodesic}

Proof. Choose an optimal map $\phi : \Gamma \to \Gamma'$. If $\Delta_\phi = \Gamma$ (and all vertices have ≥ 2 gates) the folding path is a geodesic from Γ to Γ'. If $\Delta_\phi \neq \Gamma$ start by scaling Δ_ϕ up and the edges in the complement up until the tension graph Δ increases. If there are any vertices with one gate, adjust ϕ. Continue until $\Delta = \Gamma$ and then follow with a folding path. See also [10] and [4] for further discussion. □

Exercise 34. What is the geodesic constructed in this proof in the case $\Gamma = B$ and $\Gamma' = A$ in the example in Section 2.3?

Exercise 35. Find geodesics from R to Γ and from Γ to R in Exercise 23.

Exercise 36. Can every folding path be extended forward? Can it be extended backward?

Exercise 37. Can a folding path intersect some $\Sigma(\Gamma)$ in a disconnected set?

The following three exercises finish the proof that the two metric topologies are equivalent to the length function topology.

Exercise 38. Show that for every $\Gamma \in \mathcal{X}_n$ there is $\epsilon > 0$ so that $B_\rightarrow(\Gamma, \epsilon)$ intersects only finitely many simplices. Hint: This is a consequence of local finiteness and Proposition 2.5.

Exercise 39. Show that for any Γ for a sufficiently large R the ball $B_\rightarrow(\Gamma, \epsilon)$ intersects infinitely many simplices.

Exercise 40. Show that every $B_\leftarrow(\Gamma, R)$ intersects only finitely many simplices for any $R > 0$. Thus closed backward balls are compact.

Exercise 41. Prove the equivalence of forward metric and length function topologies. Hint: That (metric)$\to$(length) is continuous is easy. For the converse use previous exercises (for which finiteness of candidates is the key).

Exercise 42. Show that the backward metric defines the topology on $\mathcal{X}_n$ equivalent to the length function topology.

Exercise 43. Length function topology is equivalent to the projectivized length function topology. Hint: The key is to show that if Γ_i are outside a neighborhood of Γ, say outside the forward ϵ-ball $B_{\to}(\Gamma,\epsilon)$, then Γ_i's cannot converge to Γ in the projective space. Let Δ_i be the point on a geodesic from Γ to Γ_i at distance ϵ from Γ. If ϵ is small the ϵ-sphere intersects only finitely many simplices and after a subsequence there is a candidate α for Γ that realizes the distance to Γ_i and there is a candidate β for graphs in the sphere with $\ell_\Gamma(\beta) > \ell_{\Delta_i}(\beta)$. Show that this implies that

$$\frac{\ell_{\Gamma_i}(\beta)}{\ell_{\Gamma_i}(\alpha)}\Big/\frac{\ell_\Gamma(\beta)}{\ell_\Gamma(\alpha)} < exp(-\epsilon)$$

finitely many length functions don't suffice (L 3).