2015

1. Find all functions \( f : \mathbb{Z} \to \mathbb{Z} \) such that the following two conditions hold:
   (i) For all \( n \in \mathbb{Z} \) we have \( f(n)f(-n) = f(n^2) \).
   (ii) For all \( m, n \in \mathbb{Z} \) we have \( f(m + n) = f(m) + f(n) + 2mn \).

2. Consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = ||x| - 1| \). Find all solutions \( x \in \mathbb{R} \) to
   \[ (f \circ f \circ \cdots \circ f)(x) = x \]
   with \( n \) a positive integer. (Note: The answer may depend on \( n \).)

3. Find all pairs of nonnegative integers \( x, y \) such that
   \[ \sqrt{x^2 + y + 1} + \sqrt{y^2 + x + 4} \]
   is an integer.

4. The two tangent lines to a circle \( C \) at points \( P \neq Q \) intersect at a point \( A \), and similarly the two
tangent lines to \( C \) at points \( P' \neq Q' \) intersect at a point \( A' \). If \( A' \) is on the line generated by \( PQ \),
   prove that \( A \) is on the line generated by \( P'Q' \).

5. Let \( A \subset \mathbb{R} \) be a finite, non-empty set of real numbers, and let \( f : A \to A \) be a function. Assume for
every \( x, y \in A \) with \( x \neq y \), it happens that \( |f(x) - f(y)| |x - y| \). Prove there exists some \( a \in A \)
such that \( f(a) = a \).

6. Find all polynomials with real coefficients \( P(x) \in \mathbb{R}[x] \) satisfying:
   \[ (x + 1)^3 P(x - 1) - (x - 1)^3 P(x + 1) = 4(x^2 - 1)P(x) \]

7. Determine
   \[ \lim_{n \to \infty} n^2 \left[ \left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n \right]. \]

2014

1. Given that \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( \int_0^1 f(x) \, dx = a \), evaluate
   \[ \int_0^1 f(x) \left( \int_0^x f(t) \, dt \right) \left( \int_x^1 f(t) \, dt \right) \, dx. \]

2. Let \( ABCD \) be a square, with side length 1. On sides \( CD \) and \( AD \) are points \( P \) and \( Q \) (respectively)
such that the perimeter of the triangle \( PDQ \) is 2. Show that the angle \( PBQ \) is 45°.

3. Consider six points in the plane, no three of which are on any given line. Thus, they determine
   fifteen segments and twenty triangles. If all the segments have different lengths, prove that there
   is a segment which is the smallest side of a triangle and the largest side of another triangle.

4. Determine
   \[ \lim_{n \to \infty} \left(1 + \frac{1}{\ln(n)}\right) \left(1 + \frac{1}{2\ln(n)}\right) \cdots \left(1 + \frac{1}{n\ln(n)}\right). \]

5. Evaluate
   \[ \int_0^\pi \frac{x \sin(x)}{1 + \cos^2(x)} \, dx. \]

6. Given integers \( n \geq 3 \) and \( 1 \leq i < j \leq n-1 \), prove that the binomial coefficients \( \binom{n}{i} \) and \( \binom{n}{j} \) are not
   relatively prime.

7. Let \( f : [0, 1] \to (0, \infty) \) be a continuous function satisfying
   \[ \int_0^1 f(x) \, dx \int_0^1 \frac{1}{f(x)} \, dx = 1. \]
   Show that \( f \) is constant.

2013
1. Five boys and five girls sit around a table. Prove that there is someone sitting between two girls.
2. Let $X, Y$ be two $n \times n$ matrices such that $XY = X + Y$. Prove that $XY = YX$.
3. A $7 \times 7$ square is tiled with ten $4 \times 1$ rectangles and one $3 \times 3$ square. What are the possible positions of the $3 \times 3$ square?
4. Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $B_1, \ldots, B_n$ be $m \times m$ matrices. Let $C$ be a block matrix, consisting of $n^2$ blocks $a_{ij}B_j$:

$$
C = \begin{bmatrix}
    a_{11}B_1 & a_{12}B_2 & \cdots & a_{1n}B_n \\
    a_{21}B_1 & a_{22}B_2 & \cdots & a_{2n}B_n \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1}B_1 & a_{n2}B_2 & \cdots & a_{nn}B_n
\end{bmatrix}
$$

Express the determinant of $C$ in terms of the determinants of $A$ and $B_1, \ldots, B_n$.
5. Let $a_1, a_2, \ldots, a_{2n}, b_1, b_2, \ldots, b_{2n}$ be non-negative real numbers such that $a_1 = a_{2n}$ and $b_1 = b_{2n}$. Prove that

$$
\min_i (a_i + b_i) \leq \sum_{i=1}^{2n-1} \min\{a_i, a_{i+1}\} + \sum_{i=1}^{2n-1} \min\{b_i, b_{i+1}\}
$$

2012

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real $x, y, z$ the relation

$$
f(f(x + y) + z) + f(x + f(y + z)) = 2y
$$

holds.
2. Triangle $ABC$ has side lengths $a$, $b$, and $c$ and median lengths $\alpha$, $\beta$, and $\gamma$. If $\alpha, \beta, \gamma$ are the side lengths of a second triangle, what are the median lengths in that triangle?
3. Determine all the real solutions of the equation

$$
(x^3 + \frac{3}{4}x)^\frac{3}{4} + \frac{3}{4}(x^3 + \frac{3}{4}x) = x.
$$

4. Prove that for every polynomial $P$ there is a polynomial $Q$ such that $Q(x^{2012})$ is a multiple of $P(x)$.
5. Find all pairs $(m, n)$ of positive integers for which $4^m + 5^n$ is a square.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $f(0) = 0$ and $f(1) = 1$. Prove that

$$
\frac{1}{e} \leq \int_0^1 |f'(x) - 2xf(x)| \, dx.
$$

7. For all real numbers $a, b, c$ consider the inequality

$$
|a - b| + |b - c| + |c - a| \leq C\sqrt{a^2 + b^2 + c^2}.
$$

(a) Prove the inequality for $C = 2\sqrt{2}$.
(b) Prove that under the additional assumption $a, b, c \geq 0$ the inequality also holds for $C = 2$.

2011

1. Evaluate the sum of the series

$$
\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}.
$$

2. Prove that if $m, n$ are positive integers such that $\sqrt{7} > \frac{m}{n}$, then $\sqrt{7} > \frac{m}{n} + \frac{1}{mn}$.
3. Solve the equation

$$
x^2 + xy + y^2 = 97
$$

for (i) natural numbers $x, y$, and (ii) integer numbers $x, y$. 

4. Prove that for any positive integers \(a, b, c, d\) the product \((a - b)(a - c)(a - d)(b - c)(b - d)(c - d)\) is divisible by 12.
5. Which positive integers can be written as the sum of \(\geq 2\) consecutive positive integers?
6. Let \(a > 0\) be a constant. Assume that \(x_0 > 0\) and
   \[
x_{n+1} = \frac{1}{3} \left( 2x_n + \frac{a}{x_n^2} \right).
   \]
   Prove that \(\lim_{n \to \infty} x_n\) exists and find it.
7. Let \(x_1, x_2, \ldots, x_n \geq 1\). Prove that
   \[
   (1 + \sqrt{x_1x_2 \cdots x_n}) \left( \frac{1}{1 + x_1} + \frac{1}{1 + x_2} + \cdots + \frac{1}{1 + x_n} \right) \geq n.
   \]

2010
1. Let \(S\) be a square. Prove that \(S\) can be divided into \(n\) squares, using line segments parallel or perpendicular to the sides of \(S\), for each integer \(n \geq 6\).
2. Evaluate the following if it exists:
   \[
   \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{nk}}
   \]
3. Let \(\Delta ABC\) be an arbitrary triangle in \(\mathbb{R}^2\) with vertices \(A, B,\) and \(C\). A frog starts from a point \(P_0 \in \mathbb{R}^2\) and travels directly toward \(A\). Upon reaching \(A\) the frog continues in the same direction to the point \(P_1\) such that \(P_0A = AP_1\). Next the frog travels from \(P_1\) directly through \(B\) to the point \(P_2\) such that \(P_1B = BP_2\). The frog then starts from \(P_2\) and travels through \(C\) to the point \(P_3\) such that \(P_2C = CP_3\). Next from \(P_3\), the frog repeats the same action with respect to \(A, B,\) and \(C\) cyclicly, generating a sequence of points \(P_1, P_2, P_3, P_4, \ldots\). What is the distance between \(P_0\) and \(P_{2010}\)?
4. Define a sequence recursively by \(x_1 = 1, x_2 = 1,\) and \(x_n = 3x_{n-1} - x_{n-2}\) for \(n > 2\). Find a closed formula for \(x_n\).
5. Prove that there is no function \(f : \mathbb{N} \to \mathbb{N}\) such that \(f(f(n)) = n + 1\) for all \(n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}\).
6. For which positive integers \(n\) can the \(n \times n\) chess board with a corner square removed be tiled by \(3 \times 1\) dominoes? For which \(n\) can the \(n \times n\) chess board with some square removed be tiled by \(3 \times 1\) dominoes?
7. A number of students sit in a circle while their teacher give them candy. Each student initially has an even number of pieces of candy. When the teacher blows a whistle, each student simultaneously gives half of his or her own candy to the neighbor on the right. Any student who ends up with an odd number of pieces of candy gets one more piece from the teacher. Show that no matter what the distribution is at the beginning, after a finite number of iterations of this transformation all students will have the same number of pieces of candy.

2009
1. Let \(S\) and \(S'\) be unit squares in \(\mathbb{R}^2\) with their centers at the origin. Find the minimum area of their intersection \(S \cap S'\). (see Fig. 1)
2. Let \(n = 3k + 1\) with \(k = 1, 2, 3, \ldots\). Consider the \(n \times n\) chess board. How many of the \(n^2\) squares \(S\) have the property that after \(S\) is removed the remaining \(n^2 - 1\) squares can be tiled by \(3 \times 1\) dominoes?
3. Let \(x, y, z \in \mathbb{R}\) satisfy \(x^2 + y^2 + z^2 = 1\) and \(x + y + z = 0\). Find \(\max(xyz)\) and \(\min(xyz)\).
4. Find an explicit real valued function \( f : \mathbb{R} \to \mathbb{R} \) (in closed form) whose Taylor series equals
\[
f(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots
\]

5. Prove that
\[
\lim_{n \to \infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k} = 2
\]

6. Let \( x_1 < x_2 < x_3 < \cdots < x_n \) be \( n \) real numbers, where integer \( n > 1 \). Prove that
\[
\sum_{i=1}^{n} \frac{1}{\prod_{j=1, j \neq i}^{n} (x_i - x_j)} = \sum_{i=1}^{n} \frac{1}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} = 0.
\]

7. Let \( f(x) : \mathbb{R}^1 \to [0, \infty) \) be continuous and differentiable. Prove
\[
\int_{0}^{t} \int_{0}^{t} f(xy) dx \, dy + \int_{0}^{t} \int_{0}^{t} xy f'(xy) dx \, dy = \int_{0}^{t} f(s) ds.
\]